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Phase separation and random domain patterns in a stochastic particle model†

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Abstract

This paper deals with a dynamics (Glauber–Kawasaki) of a d-dimensional ($d = 2, 3$) spin system, with a (zero magnetization) Bernoulli measure as initial condition. On the hydrodynamic scaling the system is reacting and diffusive, and the associated macroscopic initial state is stationary, but unstable. We prove that the system will escape from this spatially trivial state on a time scale longer than the hydrodynamic one (on this new scale the escape will happen at a deterministic time). Right after the escape the system will have locally a magnetization corresponding to one of the two stable phases, but globally it will show a nontrivial spatial structure. The onset of this spatial structure is studied and its characterization by means of a random field is given. This work extends the results in De Masi et al. (1991) that deal with a one-dimensional system.

Key words: Interacting particle systems; Reaction–diffusion equations; Correlation functions; Gaussian random field; Phase separation

1. Introduction

It has been shown (De Masi et al., 1986; Boldrighini et al., 1987, 1992) that some Glauber–Kawasaki processes give rise in the hydrodynamical limit to a behavior that can be described by reaction–diffusion equations (from now on RD equations, see (2.4)). These types of equations have been widely used (see, e.g., Fife (1979)) to study phenomena of interface formation and phase separation. In this paper we are concerned with a spin system (without external magnetic field) initially in a very high-temperature phase and then suddenly cooled down under the critical temperature. So the high-temperature phase is no longer stationary and furthermore the invariant state will be likely a superposition of more than one pure phase. The relaxation time

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for such a system is very long and essentially the system will break up in a relatively short time into regions of different magnetizations, most likely near the values of the magnetization corresponding to the pure phases. Then the regions will evolve to go toward the equilibrium. We will deal with the onset of the phase separation and we will be particularly interested in the spatial structure of these clusters.

A high-temperature phase (i.e., $m \equiv 0$) is a stationary solution for (2.4), but it is unstable. From a PDE point of view, in order to recover the physical picture we can for example add a noise, that will take into account various previously forgotten effects. Our approach is instead simply to follow the evolution of the particle system and to go to a time scale longer than the hydrodynamic one.

In a recent paper De Masi et al. (1991) solved the problem of characterizing the spatial pattern after the phase separation in a one-dimensional Glauber–Kawasaki process. They studied the system (with initial condition of zero magnetization) on long times and they showed that the system, at a certain deterministic time (with a particular time scaling), leaves the situation of zero magnetization and reaches one of the two phases (the two minima of the potential appearing in (2.4)). Obviously the mean magnetization at each point of the space cannot reach one of the two phases independent of what is happening in all the other points in the space: there is a spatial correlation such that the typical configurations of this process are characterized by alternate blocks with values of magnetization corresponding to the two phases. The particular spatial rescaling prevents us from interface problem: in this scale with probability one no point belongs to the interface. The problem in a finite volume (in which we do not see any spatial structure) has been previously solved (De Masi and Presutti, 1991b, Chapter IX). As observed in Section 2 of De Masi et al. (1991a), the right time scale of escaping can be guessed if one assumes that the system can be described by a stochastic PDE (formula (2.7) of De Masi and Presutti (1991b)), finding in this way a first reconciliation of the two different approaches.

The aim of this work is to extend the results in De Masi et al. (1991a) to the case of a system of dimension d (with the natural extension of the definition of the process and its phase space). The result (like in De Masi et al. (1991a)) is based on estimates of correlation functions of every order (see Sections 5–7). The main problem that arises in $d > 1$ is the ultraviolet divergence (logarithmical in $d = 2$ and power law in $d \geq 3$) in the spatial dependence of the correlation functions. The solution requires a careful choice of the norm in which we make the estimates; in De Masi et al. (1986, 1991a) only uniform norms were used and this approach does not extend. We will show that we have indeed a quite good control on the uniform norm of the correlation functions, but only at sufficiently long times. At shorter times we will use a weaker norm and we will prove that this is enough for our purposes. The difficulty of doing estimating at long times is given by the fact that in order to estimate a correlation function at time t, we need to have estimates at every shorter time. In Section 5 this problem is solved in the case $d = 2$ and in Section 6 in the case $d = 3$. Essentially we will get good estimates at an intermediate time t by taking advantage of the fact that, the fluctuations being increasing in time, even nonoptimal estimates at short times can be enough to get good bounds at some later time (not too long, as we shall see). We will show that t can be chosen long enough to be influenced little by the ultraviolet divergence and we will be able to extend the estimates to longer times. In the case $d = 2$ some simplifications arise, because the divergence is milder. Nevertheless the estimates we prove in this paper are sharp only for the two point correlation functions and they are in general weaker than those in De Masi et al. (1991a). However we are able to show in Sections 3 and 4 how to get results analogous to those in $d = 1$. The program is carried out only for $d = 2$ and 3, but we will sketch how to modify the proofs to let them work in higher dimensions.

Here are some general remarks:

(1) From the proofs, we will see that, when the system is really noisy, the nonlinear part is ineffective (because we have an infinitesimal magnetization) and vice versa.

(2) The problem of the ultraviolet divergence has been taken into consideration in Ravishankar (1989, 1992) addressing the problem of fluctuations from hydrodynamics for symmetric simple exclusion in $\mathbb{Z}^d$ and weakly asymmetric exclusion in $\mathbb{Z}^2$. This can be regarded as analogous to our problem in the very first moments of the evolution (our scale of time is longer than the hydrodynamical one).

(3) We will characterize the configurations after the escape (in the continuum limit) by means of a gaussian random field. In terms of lattice spacings the configuration after the escaping will vary on the scale $\varepsilon^{-1}\sqrt{|\log \varepsilon|}$ (and $\varepsilon \to 0$) and the escape takes place on times proportional to $|\log \varepsilon|$. A somewhat different problem is to study the evolution of the pattern formed soon after the escape. There is a lot of difference among systems of different dimensionality. This problem is not yet solved in one dimension, but there are some results in $d = 2, 3$ (Bonaventura, 1992). The idea is that if we take the RD equation and we make the rescaling $x = r/\sqrt{|\log \varepsilon|}$ we obtain

$$\frac{\partial}{\partial t} m(x,t) = \frac{1}{2d|\log \varepsilon|} \Delta m(x,t) - V'(m). \tag{1.1}$$

We can readily see that this type of equation has been used by Evans et al. (1991) and De Mottoni and Schatzmann (1989) to model the motion by mean curvature, so we can expect that the particle model with this space-time rescaling also will evolve following the mean curvature motion. This has been proved by Bonaventura (1992) under some conditions; essentially we must deal with only one cluster of positive magnetization, in a sea of negative magnetization and with conditions on the profile of the interface (and the result holds until the time in which the first geometric singularity appears). Even if there are various results (see, e.g., De Mottoni and Schatzmann (1989)) on the onset of interfaces, at the present it is not clear how to prove an approximate motion by mean curvature for our system after the escape.

2. The model and the main results

For all ε ($1/2 \geq \varepsilon > 0$) we consider the Markov process $(\sigma^{\varepsilon}(t))_{t\geq 0}$ taking values on $X_{\varepsilon}^d = \{-1,+1\}^{\mathbb{Z}_{\varepsilon}^d}$, where $\mathbb{Z}_{\varepsilon}^d = \mathbb{Z}^d \bmod \{[\varepsilon^{-1}|\log\varepsilon|^{1/2+\theta}]\}$ with $\theta \in (0, 1/6)$. Hence each ε-process describes a system of spins (or particles) on a d-dimensional torus. We will denote the generator of the dynamics by

$$L_{\varepsilon} = \varepsilon^{-2} L_0 + L_{\mathrm{G}}, \tag{2.1}$$

where L_0 is the generator of the simple exclusion and L_{G} is the generator of a Glauber Dynamics (spin flips) (Liggett, 1985; De Masi and Presutti, 1991b); Spohn, 1991). In detail, for all functions f defined on X_{ε}^d and for all $\sigma \in X_{\varepsilon}^d$

$$L_0 f(\sigma) = \frac{1}{2d} \sum_{x\in\mathbb{Z}_{\varepsilon}^d} \sum_{l=1}^{d} [f(\sigma^{x,x+e_l}) - f(\sigma)], \tag{2.2a}$$

$$L_{\mathrm{G}} f(\sigma) = \sum_{x\in\mathbb{Z}_{\varepsilon}^d} c(x,\sigma)[f(\sigma^x) - f(\sigma)], \tag{2.2b}$$

in which $e_l \in \mathbb{Z}^d$ is the unit vector in the l-direction. As usual

$$\sigma^{x,y}(z) = \begin{cases} \sigma(x), & \text{if } z = y, \\ \sigma(y), & \text{if } z = x, \\ \sigma(z), & \text{otherwise}, \end{cases}$$

$$\sigma^{x}(z) = \begin{cases} \sigma(z), & \text{if } z \neq x, \\ -\sigma(z), & \text{if } z = x. \end{cases}$$

We will make the choice:

$$c(\mathbf{0},\sigma) = 1 - \frac{\gamma}{d}\sigma(\mathbf{0})\left[\sum_{l=1}^{d}(\sigma(e_l) + \sigma(-e_l))\right] + \frac{\gamma^2}{d}\sum_{l=1}^{d}\sigma(+e_l)\sigma(-e_l), \tag{2.3a}$$

$$\gamma \in \left(\frac{1}{2}, 1\right), \tag{2.3b}$$

$$c(x,\sigma) = c(\mathbf{0},\tau_x\sigma) \quad (\tau_y\sigma(x) = \sigma(y+x)). \tag{2.3c}$$

We clearly see that the *flip rates*, $c(\cdot,\sigma)$, are local, traslationally invariant and strictly positive functions on X_{ε}^d.

From now on we will call $\sigma \in X_{\varepsilon}^d$ a *configuration* and $\sigma(x)$ the *value of the spin in the* x *site*. Besides we will denote by $\mathscr{M}_n^{\varepsilon}$ the set of $\underline{x} = (x_1, \ldots, x_n)$ such that $x_i \in \mathbb{Z}_{\varepsilon}^d$ for all $i = 1, \ldots, n$ and such that $x_i \neq x_j$ for all $i \neq j$.

The whole problem is to characterize the behavior of the system on long times ($\propto |\log\varepsilon|$) when the initial condition for the process is a Bernoulli measure on X_{ε}^d with zero mean in each site (i.e., each spin takes values ± 1 with probability 1/2 independent of other spins). To have a better understanding of the problem let us recall the results of an earlier paper (De Masi et al., 1986).

Let μ^ε be a product measure on $(X_\varepsilon^d, \mathscr{B}(X_\varepsilon^d))$ such that $\mu^\varepsilon(\sigma(\boldsymbol{x})) = m_\varepsilon(\varepsilon \boldsymbol{x})$, where $m_\varepsilon(\cdot)$ is a suitable regular function converging (uniformly on the compacts) to a function $m(\cdot)$ (as $\varepsilon \to 0$). It has been proved that in the limit $\varepsilon \to 0$ the correlation functions of the system factorize into the product of functions $m_\varepsilon(\cdot, t)$ ($m_\varepsilon(r,t) = \mu_t^\varepsilon(\sigma[\varepsilon^{-1} r])$ and μ_t^ε is the law of the process starting from μ^ε). Besides $m(\cdot, t) = \lim_{\varepsilon\to 0} m_\varepsilon(\cdot, t)$ exists and solves the RD equation

$$\begin{cases} \frac{\partial}{\partial t} m = \frac{1}{2d}\Delta m + F(m), \\ m(\cdot, 0) = m(\cdot), \end{cases} \tag{2.4a}$$

$$F(m) = -V'(m) = -2\nu_m(\sigma(\mathbf{0}) c(\mathbf{0}, \sigma)) \tag{2.4b}$$

in which ν_m is the Bernoulli measure such that $\nu_m(\sigma(\boldsymbol{x})) = m$. With the choice (2.3) we have:

$$V(m) = \frac{\beta}{4} m^4 - \frac{\alpha}{2} m^2, \qquad \alpha = 2(2\gamma - 1), \qquad \beta = 2\gamma^2. \tag{2.4c}$$

We indicate with $\pm m^*$ the two minima of V. In particular, formula (2.3b) implies that $m(\cdot, 0) \equiv 0$ is an unstable stationary point of (2.4). We precise that the equation (2.4) has been obtained by fixing t and letting $\varepsilon \to 0$. To see the escape from this unstable point we must go on very long times, i.e., we must let $t \to \infty$ as $\varepsilon \to 0$. A general reference for (2.4) is Smoller (1983).

Our main result is the following theorem.

Theorem 2.1. *Let μ^ε be the product measure on $(X_\varepsilon^d, \mathscr{B}(X_\varepsilon^d))$ with $\mu^\varepsilon(\sigma(\boldsymbol{x})) = 0$ for all $\boldsymbol{x} \in \mathbb{Z}_\varepsilon^d$. Set $t_f = (d/2\alpha)|\log \varepsilon| + |\log \varepsilon|^{1/3}$. For all $n \geq 1$ we have that*

$$\lim_{\varepsilon \to 0} \sup_{\underline{x} \in \mathscr{M}_n^\varepsilon} \left| \mu^\varepsilon_{\tau |\log \varepsilon|} \left(\prod_{i=1}^n \sigma(\boldsymbol{x}_i) \right) \right| = \mathbf{0}, \quad \text{if } \tau \leq \frac{d}{2\alpha} \tag{2.5a}$$

and

$$\lim_{\varepsilon \to 0} \sup_{\underline{x} \in \mathscr{M}_n^\varepsilon} \left| \mu^\varepsilon_{t_f} \left(\prod_{i=1}^n \sigma(\boldsymbol{x}_i) \right) - \tilde{\boldsymbol{E}} \left(\prod_{i=1}^n \rho(\varepsilon |\log \varepsilon|^{-1/2} \boldsymbol{x}_i) \right) \right| = \mathbf{0}, \tag{2.5b}$$

where $\rho(\boldsymbol{r}) = m^* \operatorname{sign}(\tilde{X}(\boldsymbol{r}))$. $\tilde{X}(\boldsymbol{r}) \in \mathbb{R}$ ($\boldsymbol{r} \in \mathbb{R}^d$) is the gaussian random field such that $\tilde{\boldsymbol{E}}(\tilde{X}(\boldsymbol{r})) = 0$ and $\tilde{\boldsymbol{E}}(\tilde{X}(\boldsymbol{r})\tilde{X}(\boldsymbol{r}')) = \exp\{-\alpha(\boldsymbol{r} - \boldsymbol{r}')^2/2\}$, ($\boldsymbol{r}, \boldsymbol{r}' \in \mathbb{R}^d$).

This theorem (proven for $d = 2$ and $d = 3$) shows that, on the time scale $|\log \varepsilon|$, the system escapes from zero magnetization after the deterministic time $\tau_c = d/2\alpha$. The situation soon after the escape ($t = t_f$) is described by a random field that is obtained by considering only the sign of a gaussian random field in $\mathbb{R}^d$. The length of correlation of this random field is $\propto (\alpha)^{-1/2}$ and taking into consideration the rescaling in the expectation term in (2.5b) we obtain that the scale on which varies this random field on the lattice is $\varepsilon^{-1}|\log \varepsilon|^{-1/2}/\sqrt{\alpha}$. In spite of the similarity of this result with that in De Masi et al. (1991a), it is quite clear that the spatial structure of a gaussian field in dimensions higher than one can be highly nontrivial. However the dimensionality becomes even more critical if we try to investigate the further development of the system (after t_f, more precisely for $t = \tau|\log \varepsilon|$, $\tau > \tau_c$), as we already said in the introduction.

Let $m_\varepsilon(\mathbf{r}, t; \lambda)$ be the solution of (2.4) with initial condition $m_\varepsilon(\mathbf{r}, 0; \lambda) = \lambda(\sigma([\varepsilon^{-1}\mathbf{r}]))$ and λ is a product measure on X_ε^d. For all $n \geq 1$, $\varepsilon > 0$ and $\underline{\mathbf{x}} \in \mathscr{M}_n^\varepsilon$ $(t > 0)$ we define

$$v_n^\varepsilon(\underline{\mathbf{x}}, t; \lambda) \equiv \mathbf{E}_\lambda^\varepsilon\left(\prod_{i=1}^n [\sigma(\mathbf{x}_i, t) - m_\varepsilon(\varepsilon \mathbf{x}_i, t; \lambda)] \right). \tag{2.6}$$

Furthermore if $\lambda = \mu^\varepsilon$ (zero average product measure) we shall write

$$v_n^\varepsilon(\underline{\mathbf{x}}; t) \equiv v_n^\varepsilon(\underline{\mathbf{x}}, t; \mu^\varepsilon) = \mathbf{E}_{\mu^\varepsilon}^\varepsilon\left(\prod_{i=1}^n \sigma(\mathbf{x}_i, t) \right). \tag{2.7}$$

By symmetry $v_{2n+1}^\varepsilon(\mathbf{x}; t) \equiv 0$ for all $n \in \mathbb{Z}^+$.

I will present two somewhat different proofs in two and three dimensions. In the bidimensional case the divergence of the correlation functions at the origin (in the spatial dependence) is only logarithmic and we will show that we can obtain Theorem 2.1 by using only the *rough* estimates on correlation functions of order greater than two (given by the Theorem 5.2). The case $d \geq 3$ is somewhat more complex, because the divergence of the correlation functions has a power behavior and we must choose carefully the norm with which we make the estimate (the divergence is integrable, so if we make estimates on some integral expression we can avoid divergences). On the other side we shall see that the divergence vanishes on long times.

As in De Masi et al. (1991a), we will distinguish two phases in the escaping: the early stage and the final stage. For further details see De Masi et al. (1991a).

3. The early stage of escape

This section is analogous to Section 3 of De Masi et al. (1991a). At this level the real modifications are concerned with the proof of tightness that requires nonuniform estimates on the correlation functions.

For all $\phi \in \mathscr{S}(\mathbb{R}^d)$, $t \geq 0$ and $\varepsilon > 0$ we define

$$Y_t^\varepsilon(\phi) = \delta^{d/2} \exp(-\alpha t) \sum_{\mathbf{x}} \phi(\delta \mathbf{x}) \sigma(\mathbf{x}, t) \quad \left(\delta = \frac{\varepsilon}{\sqrt{|\log \varepsilon|}} \right) \tag{3.1a}$$

and for $\tau < d/2\alpha$

$$X_\tau^\varepsilon(\phi) = Y_{\tau|\log \varepsilon|}^\varepsilon(\phi). \tag{3.1b}$$

By so doing we have taken into account the growth of the fluctuations up to $t = \tau_c|\log\varepsilon|$. We are able to prove that this rescaled field converges in law to an Ornstein Uhlenbeck process. To be more precise, the path space will be alternatively $\Omega = C([0,\infty), \mathscr{S}'(\mathbb{R}^d))$ or $\Omega = D([0,\infty), \mathscr{S}'(\mathbb{R}^d))$. $\{X_\tau(\phi), \phi \in \mathscr{S}(\mathbb{R}^d)\}_{\tau \geq 0}$ is the coordinate process on $\Omega(X_\tau(\phi)(\omega) = \omega(\tau)(\phi)$, for all $\omega \in \Omega)$.

Consider now the space $C([0,\infty), \mathscr{S}'(\mathbb{R}^d))$ and a law $\mathscr{P}$ on it which is concentrated on the deterministic evolution

$$X_\tau(\phi) = X_0(\phi_\tau), \tag{3.2a}$$

where

$$\phi_\tau(y) = \int_{\mathbb{R}^d} \mathrm{d}z\, \phi(z) \frac{1}{(2\pi\tau/d)^{d/2}} \exp\left\{ -\frac{(z-y)^2}{2\tau/d} \right\} \tag{3.2b}$$

and under $\mathscr{P}$ X_0 is Gaussian with

$$\mathscr{E}(X_0(\phi)) = 0, \qquad \mathscr{E}(X_0(\phi)X_0(\psi)) = \left(1 + \frac{2}{\alpha}\right)\int_{\mathbb{R}^d} \mathrm{d}x\, \phi(x)\psi(x) \tag{3.3}$$

for all ϕ and ψ in $\mathscr{S}(\mathbb{R}^d)$. Furthermore we will call $\mathscr{P}^\varepsilon$ the law of the process $X_\tau^\varepsilon(\phi)$ $(\tau \geq 0)$ on the space $D([0,\infty), \mathscr{S}'(\mathbb{R}^d))$. This law is naturally induced by the original ε-process with starting measure μ^ε. Now we can state the following theorem.

Theorem 3.1. *For each $\bar\tau$ and $\tau_0 \in (0, \tau_c)$ with $\tau_0 < \bar\tau$ $(\tau_c = d/2\alpha)$, the law $\mathscr{P}^\varepsilon$ restricted to $D([\tau_0, \bar\tau), \mathscr{S}'(\mathbb{R}^d))$ converges weakly to the restriction of $\mathscr{P}$ to $D([\tau_0, \bar\tau), \mathscr{S}'(\mathbb{R}^d))$.*

For some comments on this result see Section 3 of De Masi et al. (1991a). Here we simply observe that X_0^ε converges weakly in $\mathscr{S}'(\mathbb{R}^d)$ to the standard white noise (by the central limit theorem). So the result of Theorem 3.1 cannot be extended to $D([0, \bar\tau)$, $\mathscr{S}'(\mathbb{R}^d))$ (in (3.3) there is the factor $(1 + 2/\alpha)$ instead of 1). This is due to the time rescaling we adopted ($|\log\varepsilon|$): what happens on shorter time scales (essentially on finite times) is shrunk in a point ($t = 0$) and reveals itself as a *jump at time zero*. We see that the greater the linear instability, the smaller is the jump at time zero: this can be explained by observing that the jump is due to noise effects which are in competition with deterministic drift given by the linear instability.

Proof of Theorem 3.1. The line of the proof is identical to that in De Masi et al. (1991a) and so I refer to it for a more complete explanation. Here I will give in detail the proof of the *tightness* and a sketch of the rest.

Following the discussion after the statement of the Theorem, we have three main steps: (i) estimates on times $\tau|\log\varepsilon|$ (tightness for the family of measures we are going to define and identification of the limit), (ii) estimates on finite times and (iii) connection between these two scales.

Concerning the tightness in (i) we define

$$\Delta_\delta^d \varphi(\delta x) = \sum_{l=1}^{d} \left(\frac{\varphi(\delta x + \delta e_l) + \varphi(\delta x - \delta e_l) - 2\varphi(\delta x)}{\delta^2} \right), \tag{3.4}$$

$$|\nabla_\delta^d \varphi(\delta x)|^2 = \sum_{l=1}^{d} \left(\frac{\varphi(\delta x + \delta e_l) - \varphi(\delta x)}{\delta} \right)^2 \tag{3.5}$$

and so we have that

$$\gamma_1^\varepsilon(\tau, \phi) \equiv |\log \varepsilon| \{ L_\varepsilon X_\tau^\varepsilon(\phi) - \alpha X_\tau^\varepsilon(\phi) \}$$

$$= (1 + 4\gamma\varepsilon^2)\delta^{d/2} e^{-\alpha t} \frac{1}{2d} \sum_x \Delta_\delta^d \phi(\delta x) \sigma(x, t) \tag{3.6_1}$$

$$- \frac{2\gamma^2}{d} |\log \varepsilon| \delta^{d/2} e^{-\alpha t} \sum_x \sum_{l=1}^{d} \sigma(x, t)\sigma(x - e_l, t)\sigma(x + e_l, t)\phi(\delta x), \tag{3.6_2}$$

$$\gamma_2^\varepsilon(\tau, \phi) \equiv |\log \varepsilon| \{ L_\varepsilon X_\tau^\varepsilon(\phi)^2 - 2X_\tau^\varepsilon(\phi) L_\varepsilon X_\tau^\varepsilon(\phi) \}$$

$$= e^{-2\alpha t} \frac{1}{2d} \delta^d \sum_x \sum_{l=1}^{d} |\nabla_\delta^d \phi(\delta x)|^2 [\sigma(x, t) - \sigma(x + e_l, t)]^2 \tag{3.7_1}$$

$$+ 4|\log \varepsilon| \delta^d e^{-2\alpha t} \sum_x c(x, \sigma(\cdot, t)) \phi^2(\delta x) \tag{3.7_2}$$

($t = \tau|\log \varepsilon|$ in the last two formulae).

Regarding (ii) (finite times) we know that the fluctuation field to take into consideration is $Y_t^\varepsilon(\phi)$, defined by (3.1a). Let

$$\bar{\gamma}_1^\varepsilon(t, \phi) = L_\varepsilon Y_t^\varepsilon(\phi) - \alpha Y_t^\varepsilon(\phi), \tag{3.8}$$

$$\bar{\gamma}_2^\varepsilon(t, \phi) = L_\varepsilon Y_t^\varepsilon(\phi)^2 - 2Y_t^\varepsilon(\phi) L_\varepsilon Y_t^\varepsilon(\phi), \tag{3.9}$$

(the explicit expressions are the same as (3.6) and (3.7) divided by $|\log \varepsilon|$). These definitions are justified by the fact that

$$X_\tau^\varepsilon(\phi) - X_{\tau_0}^\varepsilon(\phi) - \int_{\tau_0}^{\tau} ds\, \gamma_1^\varepsilon(s, \phi) \equiv M_{\tau,\tau_0}^\varepsilon(\phi), \tag{3.10}$$

$$M_{\tau,\tau_0}^\varepsilon(\phi)^2 - \int_{\tau_0}^{\tau} ds\, \gamma_2^\varepsilon(s, \phi) \tag{3.11}$$

(resp. analogous formulae hold for $Y_t^\varepsilon(\phi)$, with integrals between 0 and t) are martingales vanishing at $\tau = \tau_0$.

What we want to prove is the tightness on $D([\tau_0, \bar{\tau}), \mathscr{S}'(\mathbb{R}^d))$ for the family $\{\mathscr{P}^\varepsilon\}_{\varepsilon>0}$ ($\mathscr{P}^\varepsilon$ is the law of $X_\tau^\varepsilon(\phi)$, $\tau \in [\tau_0, \bar{\tau})$) and the tightness on $D([0, t), \mathscr{S}'(\mathbb{R}^d))$ of $\{\boldsymbol{P}^\varepsilon\}_{\varepsilon>0}$ ($\boldsymbol{P}^\varepsilon$ is the law of $Y_s^\varepsilon(\phi)$, $s \in [0, t]$). We are dealing with Poisson processes and so we need to verify

$$\sup_{\tau_0 \le \tau \le \bar{\tau}} \boldsymbol{E}_{\mu^\varepsilon}^\varepsilon (X_\tau^\varepsilon(\phi)^2 + \gamma_1^\varepsilon(\tau, \phi)^2 + \gamma_2^\varepsilon(\tau, \phi)^2) \le c, \tag{3.12}$$

$$\sup_{0 \le s \le t} \boldsymbol{E}_{\mu^\varepsilon}^\varepsilon (Y_\tau^\varepsilon(\phi)^2 + \bar{\gamma}_1^\varepsilon(t, \phi)^2 + \bar{\gamma}_2^\varepsilon(t, \phi)^2) \le c \tag{3.13}$$

and we have more than the tightness, in fact we have that the sequence $\{\mathscr{P}^\varepsilon\}_{\varepsilon > 0}$ (resp. $\{\boldsymbol{P}^\varepsilon\}_{\varepsilon > 0}$) has a convergent subsequence to a law concentrated on $C([\tau_0, \bar{\tau}), \mathscr{S}'(\mathbb{R}^d))$ (resp. $C([0, t), \mathscr{S}'(\mathbb{R}^d))$). For a proof of this fact we refer to the work of De Masi and Presutti (1991b, Chapter II and references therein).

As already said, the proof of the tightness ((3.12) and (3.13)) in dimensions higher than one contains some nontrivial problems connected with the divergences of correlation functions. This point is completely carried out only in two and three dimensions, although I believe that the method works also in higher dimensions.

Remark. We must show that the expectations of the square of some terms are bounded. As we can see from (3.6) and (3.7) these terms are written as sum of some other terms: obviously it is sufficient to show that the expectation of the square of each addendum is bounded.

Now we start with the proof of (i) and (ii). The proof of Theorem 3.1 in the bidimensional case is very similar to the one in one dimension; this is simply because Theorem 5.2 is not very far from optimality in the bidimensional case. Nevertheless we will encounter some new problems that will be the same as in the 3D case, that is why we will deal with the two cases (2 and 3 dimensions) simultaneously as far as possible. Besides we will also consider the cases (i) (3.12) and (ii) (3.13) together: in the formulae an index i will appear which takes value 0 in case (i) and 1 in case (ii).

We start with the terms in (3.6) (and the analogous for $\bar{\gamma}_1^\varepsilon$ defined by (3.8)). So we have to prove that

$$\frac{1}{|\log \varepsilon|^{2i}} \delta^d \mathrm{e}^{-2\alpha s} \frac{1}{d^2} \sum_{x \ne y} \Delta_\delta^d \phi(\delta x) \Delta_\delta^d \phi(\delta y) v_2^\varepsilon(x, y; s) \tag*{$(3.14)_1$}$$

$$+ \frac{1}{|\log \varepsilon|^{2i}} \delta^d \mathrm{e}^{-2\alpha s} \frac{1}{d^2} \sum_x (\Delta_\delta^d \phi(\delta x))^2 \tag*{$(3.14)_2$}$$

is bounded by a constant $c(\phi)$. In this formula $d = 2$ or 3 and $s \in [\tau_0 |\log \varepsilon|, \bar{\tau} |\log \varepsilon|)$ or $s \in [0, t)$ according to the case. It is easy to see that this is true because of Theorem 5.4 (in $d = 2$) or because of Theorem 6.2 (in $d = 3$) for $(3.14)_1$. The term $(3.14)_2$ is trivially bounded.

Concerning $(3.6)_2$ it is sufficient to prove

$$\gamma^4|\log\varepsilon|^{2-2i}\mathrm{e}^{-2\alpha s}\left(\sum_{x,y}|\phi(\delta x)\phi(\delta y)|/\mathbf{1}_{(|x-y|\leq 2)}\right) \tag{$(3.15)_1$}$$

$$+\frac{\gamma^4|\log\varepsilon|^{2-2i}\mathrm{e}^{-2\alpha s}\delta^d}{\mathrm{d}^2}\sum_{l,l'}\sum_{|x-y|>2}\phi(\delta x)\phi(\delta y)v_6^\varepsilon(x,x+e_l,$$

$$x-e_l,y,y+e_{l'},y-e_{l'};s) \tag{$(3.15)_2$}$$

is bounded. The term $(3.15)_1$ is bounded by

$$\gamma^4(|\log\varepsilon|^2\varepsilon^{2\alpha\tau_0})^{1-i}(5^d\textstyle\sum(\phi)+\mathrm{O}(\delta))\leq C(\phi). \tag{3.16}$$

In Formula (3.16) $\sum(\phi)\equiv\delta^d\sum_x\phi(\delta x)^2$. Observe that we need $s\geq\tau_0|\log\varepsilon|$ and $\tau_0>0$. Now we consider $(3.15)_2$. In the 2D case by Theorem (5.2) we have

$$|\log\varepsilon|^{2-2i}\mathrm{e}^{-2\alpha s}\delta^2\sup_{l,l'}\sum_{|x-y|>2}|\phi(\delta x)\phi(\delta y)|v_6^\varepsilon(x,x+e_l,x-e_l,y,y+e_{l'},y-e_{l'};s)$$

$$\leq\left(\delta^4\sum_{x,y}|\phi(\delta x)\phi(\delta y)|\right)c_6\varepsilon^{4-6b}\mathrm{e}^{4\alpha s}|\log\varepsilon|^{3-2i} \tag{3.17}$$

and the result follows because we can choose b as small as we want (if $i=1$ it is trivial, otherwise use $s\leq\bar{\tau}|\log\varepsilon|$ and $\bar{\tau}<\tau_c$). In three dimensions we must be a little careful. In this case we will consider separately the two cases $s<\tilde{\tau}|\log\varepsilon|$ (that will include $i=1$) and $s\geq\tilde{\tau}|\log\varepsilon|$ ($\tilde{\tau}>\tau_c/3$ as in the Theorem 6.3). So for $(3.15)_2$ we have the bounds:

$$c(\phi)|\log\varepsilon|^{2-2i+3/2}\mathrm{e}^{4\alpha s}\varepsilon^{3-6b},\quad s\in[0,t)\cup[\tau_0|\log\varepsilon|,\tilde{\tau}|\log\varepsilon|) \tag{3.18}$$

by Theorem 5.2 and ($i=0$)

$$c(\phi)|\log\varepsilon|^{2+3/2}(\mathrm{e}^{4\alpha s}\varepsilon^{6-3\xi})\quad(s\in[\tilde{\tau}|\log\varepsilon|,\bar{\tau}|\log\varepsilon|)) \tag{3.19}$$

by Theorem 6.3. The result follows from the observation that we can take $\bar{\tau}<\tau_c/2$ and from the fact that b and ξ can be arbitrarily chosen (the only restriction is that they must be strictly positive).

Implicit in the preceding proof is the proof of

$$\sup_{\tau\leq\bar{\tau}}\boldsymbol{E}_{\mu^\varepsilon}^\varepsilon(X_\tau^\varepsilon(\phi)^2)<c,$$

$$\sup_{0\leq s<t}\boldsymbol{E}_{\mu^\varepsilon}^\varepsilon(Y_s^\varepsilon(\phi)^2)<c \tag{3.20}$$

(same computations as for $(3.14)_1$). So we are left with $\gamma_2^\varepsilon(\bar{\gamma}_2^\varepsilon)$ (formula (3.7)). For $(3.7)_1$ observe that $[\sigma(x)-\sigma(x+e_l)]^2\leq 4$ and the result is immediate. Furthermore we have that the expectation of the square $(3.7)_2$ is bounded simply because the term itself is bounded by

$$8(|\log\varepsilon|^i e^{-2\alpha s})\left(\delta^d \sum_x \phi^2(\delta x)\right) \tag{3.21}$$

because $|c(x,\sigma)| \le 2$ (remember that $s \ge \tau_0 |\log\varepsilon|$ if $i = 0$).

The rest of the proof (point (iii)) does not differ significantly from that in De Masi et al. (1991a) and we will give only a fast sketch. First of all we can choose a sequence ε_k such that $\mathscr{P}^{\varepsilon_k} \to \mathscr{P}$ and we denote by $X_\tau(\cdot)$ the canonical process under $\mathscr{P}$ (the limit must concentrate on $C([\tau_0, \bar{\tau}), \mathscr{S}'(\mathbb{R}))$. From (3.6) and (3.10) we get that

$$X_\tau(\phi) - X_{\tau_0}(\phi) - \int_{\tau_0}^{\tau} \mathrm{d}\tau'\, X_{\tau'}\left(\frac{1}{2d}\Delta\phi\right) = M_{\tau,\tau_0}(\phi) \tag{3.22}$$

is a martingale (and $M_{\tau_0,\tau_0}(\phi) = 0$) and by (3.7) and (3.11) M^2_{τ,τ_0} is also a martingale. This implies $M_{\tau,\tau_0} = 0$ with probability one and so by (3.22) we get

$$X_\tau(\phi) = X_{\tau_0}(\phi_{\tau-\tau_0}) \tag{3.23}$$

(ϕ_τ is defined in (3.2b)), i.e., the evolution is concentrated on the deterministic *heat* evolution. By a diagonalization procedure we see that we can take τ_0 arbitrarily small. So in some sense what we need to know is X_0. More precisely from (3.6), (3.7), (3.10) and (3.11), by using Cauchy Schwartz and Theorem (5.4) (or (6.2)) we get

$$\boldsymbol{E}^{\varepsilon}_{\mu^\varepsilon}([X^{\varepsilon}_{\tau_0}(\phi) - X^{\varepsilon}_{s_0/|\log\varepsilon|}(\phi)]^2) \le c(\tau_0 + e^{-2\alpha s_0}) \tag{3.24}$$

so we can use (3.24) to connect the two time scales ($t \sim 1$ and $t \sim |\log\varepsilon|$: let $\tau_0 \to 0$ and $s_0 \to \infty$). On the time scale $t \sim 1$, $\boldsymbol{P}^\varepsilon \to \boldsymbol{P}$ (probability measure on $C([0,t], \mathscr{S}'(\mathbb{R}))$ along subsequences. In this case we get that under $\boldsymbol{P}$ the canonical process $X_t(\cdot)$ is gaussian with

$$\boldsymbol{E}(X_0(\phi)) = 0, \qquad \boldsymbol{E}(X_0(\phi)X_0(\Psi)) = \int_{\mathbb{R}^d} \mathrm{d}r\, \phi(r)\psi(r) \tag{3.25}$$

for all ϕ, ψ in $\mathscr{S}(\mathbb{R})$. By (3.8), by the analog of (3.10) and (3.11) and by the Gibbs Boltzmann principle (De Masi et al., 1986) we get that

$$X_t(\phi)^2 - 4\int_0^t \mathrm{d}s\, e^{-2\alpha s}\int_{\mathbb{R}^d} \mathrm{d}r\,\phi(r)^2 \tag{3.26}$$

is a martingale. This uniquely determines the limit. Clearly the equal time correlation kernel of $\boldsymbol{P}$ is

$$C(r, r', t) = \delta(r - r')\left[1 + \frac{2}{\alpha}(1 - e^{-2\alpha t})\right]. \tag{3.27}$$

By merging together (3.23), (3.24) and (3.26) we conclude. □

4. The final stage of escape: The proof of Theorem 2.1

Provided we have the estimates on the correlation functions, all the arguments used in this section work regardless of the dimension of the space. Also in this section, the proofs are carried over only in the cases $d = 2$ and 3. With respect to the content in De Masi et al. (1991a), the proof is changed both for the natural analytical complications that arise in more than one dimension and for the fact that in this case we have a weaker result on the correlation functions (compare results in Section 5 of De Masi et al. (1991a) with Theorem (6.3) here).

The idea of the proof is the following: we know by Theorems (5.2) and (6.3) that we have propagation of chaos with respect to $m = 0$ (up to a certain time); this already tells us that the system stays in the unstable state for all $t \leq \tau|\log \varepsilon|$ ($\tau < \tau_c$). The questions are: what happens near $\tau_c|\log \varepsilon|$? What happens (right) after $\tau_c|\log \varepsilon|$? To solve this problem we will put ourselves sufficiently near t_c, in order to be able to apply Theorem 5.1 that says that our random field is in any case well described by $m(\boldsymbol{r}, t)$ (Solution of (2.4) with suitable initial condition), at least for a short time. So we fix $a > 0$ such that αa is very small ($\alpha a < 1/30$) and $3a < a^*$, a^* as in Theorem 5.1. We now define the following reference times:

$$t^* = t_c - 2t_a, \qquad t_a = a|\log \varepsilon|, \qquad t_f = t_c + |\log \varepsilon|^{1/3} \tag{4.1}$$

and we have to study the initial value problem:

$$\begin{cases} \frac{\partial}{\partial t} m_\varepsilon(\boldsymbol{r}, t) = \frac{1}{2d}\Delta m_\varepsilon(\boldsymbol{r}, t) + \alpha m_\varepsilon(\boldsymbol{r}, t) - \beta m_\varepsilon^3(\boldsymbol{r}, t), \\ m_\varepsilon(\boldsymbol{r}, 0) = \sigma^*([\varepsilon^{-1}\boldsymbol{r}]) \end{cases} \tag{4.2}$$

in which σ^* will be the configuration at time t^*. But we already know that the magnetization, for times shorter than the critical one (t_c), vanishes in ε. Hence it seems very reasonable to start studying the linearized system

$$\begin{cases} \frac{\partial}{\partial t} l_\varepsilon(\boldsymbol{r}, t) = \frac{1}{2d}\Delta l_\varepsilon(\boldsymbol{r}, t) + \alpha l_\varepsilon(\boldsymbol{r}, t), \\ l_\varepsilon(\boldsymbol{r}, 0) = \sigma^*([\varepsilon^{-1}\boldsymbol{r}]). \end{cases} \tag{4.3}$$

Given $f: \mathbb{R}^d \to \mathbb{R}$, set $\|f(\cdot)\| = \sup_{\boldsymbol{r}}|f(\boldsymbol{r})|$. Moreover throughout this section $|\boldsymbol{r}| = \max_{1 \leq i \leq d}|r_i|$ ($\boldsymbol{r} \in \mathbb{R}^d$).

Proposition 4.1. (Bound on the linear evolution). *For any* $\eta \in (d/12, (1/4 - \theta)d)$ ($d = 2, 3$) *there are strictly positive constants* c *and* $\bar{u}$ *such that for every* $t \in [0, 2t_a)$

$$\boldsymbol{P}^\varepsilon_{\mu^\varepsilon}\left(\left\{\|l_\varepsilon(\cdot, t; \delta_{\sigma^*})\| > \frac{\varepsilon^{2\alpha a} e^{\alpha t}}{|\log \varepsilon|^\eta}\right\}\right) \leq c_1 |\log \varepsilon|^{-\bar{u}} \tag{4.4}$$

in which $\{\ldots\}$ *stands for* $\{\sigma^*: \ldots\}$ *and when we compute the probability,* σ^* *is the configuration at time* t^* *(remember that* θ *is chosen in Section 2 and it is connected with the volume of the space).*

Proof. Choose $\eta' \in (\eta, d(1/4-\theta))$ and define $A \subset X_\varepsilon^d$

$$A \equiv \left\{ \sup_{|r| \le |\log \varepsilon|^{1/2-\theta}/2} |l_\varepsilon(\mathbf{0}, t; \delta_{\sigma^*}) - l_\varepsilon(\boldsymbol{r}, t; \delta_{\sigma^*})| < \frac{\varepsilon^{2\alpha a} e^{\alpha t}}{|\log \varepsilon|^{\eta'}} \right\} \tag{4.5}$$

in which we used the fact that l_ε is periodic (σ^* is defined on a torus). We can write l_ε explicitly

$$l_\varepsilon(\boldsymbol{r}, t; \delta_{\sigma^*}) = \int d\boldsymbol{r}' \, \mathscr{G}_t(\boldsymbol{r} - \boldsymbol{r}') e^{\alpha t} \sigma^*([\varepsilon^{-1} \boldsymbol{r}']) \tag{4.6}$$

in which $\mathscr{G}_t(\boldsymbol{r}) = 2\pi t/d)^{-d/2} \exp(-\boldsymbol{r}^2/(2t/d))$. We have that

$$\begin{aligned} &\boldsymbol{P}^\varepsilon_{\mu^\varepsilon}\left(\left\{ \| l_\varepsilon(\cdot, t; \delta_{\sigma^*}) \| > \frac{\varepsilon^{2\alpha a} e^{\alpha t}}{|\log \varepsilon|^\eta} \right\} \right) \\ &\quad \le \boldsymbol{P}^\varepsilon_{\mu^\varepsilon}\left(\left\{ \| l_\varepsilon(\cdot, t; \delta_{\sigma^*}) \| > \frac{\varepsilon^{2\alpha a} e^{\alpha t}}{|\log \varepsilon|^\eta} \right\} \cap A \right) + \boldsymbol{P}^\varepsilon_{\mu^\varepsilon}(A^c). \end{aligned} \tag{4.7}$$

So we have to estimate two terms. Regarding the first one (once fixed $p \in (0,1)$) for $\varepsilon \in (0,1)$ sufficiently small we have

$$\boldsymbol{P}^\varepsilon_{\mu^\varepsilon}\left(\left\{ \| l_\varepsilon(\cdot, t; \delta_{\sigma^*}) \| > \frac{\varepsilon^{2\alpha a} e^{\alpha t}}{|\log \varepsilon|^\eta} \right\} \cap A \right) \le \boldsymbol{P}^\varepsilon_{\mu^\varepsilon}\left(\left\{ |l_\varepsilon(\mathbf{0}, t; \delta_{\sigma^*})| > \frac{p\varepsilon^{2\alpha a} e^{\alpha t}}{|\log \varepsilon|^\eta} \right\} \cap A \right) \tag{4.8}$$

and obviously we can bound the last term replacing A with the whole space. By using Chebyshev inequality and Theorem 5.3 ($d=2$) or Theorem 6.1 ($d=3$) (remember that $t^* > \tilde{\tau}|\log \varepsilon|$ in the tridimensional case, see Theorem 6.1) we have:

$$\begin{aligned} &\boldsymbol{P}^\varepsilon_{\mu^\varepsilon}\left(\left\{ |l_\varepsilon(\mathbf{0}, t; \delta_{\sigma^*})| > \frac{p\varepsilon^{2\alpha a} e^{\alpha t}}{|\log \varepsilon|^\eta} \right\} \right) \\ &\quad \le \frac{p^{-2} \varepsilon^{-4\alpha a} e^{-2\alpha t}}{|\log \varepsilon|^{-2\eta}} \int\!\!\int d\boldsymbol{r} \, d\boldsymbol{r}' \, \mathscr{G}_t(\boldsymbol{r}) \mathscr{G}_t(\boldsymbol{r}') e^{2\alpha t} \boldsymbol{E}^\varepsilon_{\mu^\varepsilon}(\sigma^*([\varepsilon^{-1}\boldsymbol{r}]) \sigma^*([\varepsilon^{-1}\boldsymbol{r}'])) \\ &\quad \le \frac{p^{-2} \varepsilon^{-4\alpha a}}{|\log \varepsilon|^{-2\eta}} \int\!\!\int d\boldsymbol{r} \, d\boldsymbol{r}' \, \mathscr{G}_t(\boldsymbol{r}) \mathscr{G}_t(\boldsymbol{r}') \left[\mathbf{1}_{(|\boldsymbol{r}-\boldsymbol{r}'| \le \varepsilon)} + \mathbf{1}_{(|\boldsymbol{r}-\boldsymbol{r}'| > \varepsilon)} \frac{c e^{2\alpha t^*} \varepsilon^d}{|\log \varepsilon|^{d/2}} \right] \end{aligned} \tag{4.9}$$

in fact if $\boldsymbol{r} - \boldsymbol{r}'| \le$, $\sigma^*([\varepsilon^{-1}\boldsymbol{r}])\sigma^*([\varepsilon^{-1}\boldsymbol{r}']) = 1$. By observing that $\varepsilon^{-4\alpha a} e^{2\alpha t^*} \varepsilon^d = 1$ and that the integral in the region $|\boldsymbol{r} - \boldsymbol{r}'| \le \varepsilon$ is bounded by $c\varepsilon^d$ we get that there is c such that

$$\boldsymbol{P}^\varepsilon_{\mu^\varepsilon}\left(\left\{ |l_\varepsilon(\mathbf{0}, t; \delta_{\sigma^*})| > \frac{p\varepsilon^{2\alpha a} e^{\alpha t}}{|\log \varepsilon|^\eta} \right\} \right) \le \left(\frac{c}{p^2} \right) \frac{1}{|\log \varepsilon|^{-2\eta + d/2}}. \tag{4.10}$$

Let us turn to the second term in the right-hand side of (4.7). We have that

$$\boldsymbol{P}^\varepsilon_{\mu^\varepsilon}(A^c) \le \boldsymbol{P}^\varepsilon_{\mu^\varepsilon}\left(\left\{ D^{(d)}_{1/2|\log \varepsilon|^{1/2+\theta}}(\mathbf{0}, l_\varepsilon(\cdot, t; \delta_{\sigma^*})) \ge \frac{\varepsilon^{\alpha a} e^{2\alpha t}}{|\log \varepsilon|^{2\eta'}} \right\} \right) \tag{4.11}$$

in which ($d = 2$)

$$D_L^{(2)}(r, f) \equiv 6L\left[\sum_{i=1,2}\int_{r_i-L}^{r_i+L}\left(\frac{\partial f}{\partial r_i''}(r'')\right)^2 \mathrm{d}r_i'' + 2L\int_{r_1-L}^{r_1+L}\int_{r_2-L}^{r_2+L} \mathrm{d}r_1''\,\mathrm{d}r_2'' \times\left(\frac{\partial^2 f}{\partial r_1''\partial r_2''}(r'')\right)^2\right] \geq \sup_{|r-r'|\leq L} |f(r) - f(r')|^2 \tag{4.12a}$$

that is defined for all $f:\mathbb{R}^2 \to \mathbb{R}$, $f \in C^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, for all $r \equiv (r_1, r_2) \in \mathbb{R}^2$, $L \in \mathbb{R}^+$ and in which the component of r'' with respect to which there is no derivation is equal to the respective component of r. Analogously ($d = 3$) for all $f:\mathbb{R}^3 \to \mathbb{R}$, $f \in C^3(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ and for all $r \in \mathbb{R}^3$, $L \in \mathbb{R}^+$

$$D_L^{(3)}(r, L) \equiv 14L\left[\sum_{i=1}^{3}\int_{r_i-L}^{r_i+L}\left(\frac{\partial f}{\partial r_i''}(r_i'')\right)^2 \mathrm{d}r_i'' + 2L\sum_{i,j}\int_{r_i-L}^{r_i+L}\int_{r_j-L}^{r_j+L} \times\left(\frac{\partial^2 f}{\partial r_i''\partial r_j''}(r_i'')\right)^2 \mathrm{d}r_i\,\mathrm{d}r_j + 4L^2\int_{r_1-L}^{r_1+L}\int_{r_2-L}^{r_2+L}\int_{r_3-L}^{r_3+L} \times\left(\frac{\partial^3 f}{\partial r_1''\partial r_2''\partial r_3''}(r'')\right)^2 \mathrm{d}r_1''\,\mathrm{d}r_2''\,\mathrm{d}r_3''\right] \geq \sup_{|r-r'|\leq L} |f(r) - f(r')|^2 \tag{4.12b}$$

with the analogous convention for r'' as in (4.12a).

The inequalities in (4.12a) and (4.12b) are clearly Sobolev-type inequalities. Here is the sketch of the proof. For $d = 2$ we have

$$f(r) - f(r') = \int_{r_1'}^{r_1}\frac{\partial f}{\partial r''}(r'', r_2)\,\mathrm{d}r'' + \int_{r_2'}^{r_2}\frac{\partial f}{\partial r''}(r_1, r'')\,\mathrm{d}r'' + \int_{r_2'}^{r_2}\int_{r_1}^{r_1'}\frac{\partial^2 f}{\partial r_1''\partial r_2''}(r_1'', r_2'')\,\mathrm{d}r_1''\,\mathrm{d}r_2''$$

by squaring and applying the Cauchy Schwartz inequality we get the bound (4.12a). Analogously for (4.12b).

By using Chebyshev inequality in formula (4.11) (let us restrict to the case $d = 2$, the case $d = 3$ is analogous)

$$\begin{aligned} P_{\mu^\varepsilon}^\varepsilon(A^c) \leq{}& \frac{\varepsilon^{-4\alpha a}}{|\log\varepsilon|^{-2\eta'}} e^{2\alpha t} 3|\log\varepsilon|^{1/2+\theta}\Bigg\{\sum_{i=1,2}\int \mathrm{d}r_i \int\!\!\int \mathrm{d}r'\,\mathrm{d}r''\,\frac{e^{2\alpha t}}{t}\mathscr{G}_t^{(i)}(|r' - r_i|) \\ &\times \mathscr{G}_t^{(i)}(|r'' - r_i|) E_{\mu^\varepsilon}^\varepsilon(\sigma^*([\varepsilon^{-1}r'])\sigma^*([\varepsilon^{-1}r''])) + |\log\varepsilon|^{1/2+\theta}\int\!\!\int \mathrm{d}r_1\,\mathrm{d}r_2 \\ &\times \int\!\!\int \mathrm{d}r'\,\mathrm{d}r''\,\frac{e^{2\alpha t^*}}{t^2}\mathscr{G}_t^{(1,2)}(|r' - (r_1, r_2)|)\mathscr{G}_t^{(1,2)}(|r'' - (r_1, r_2)|) \\ &\times E_{\mu^\varepsilon}^\varepsilon(\sigma^*([\varepsilon^{-1}r'])\sigma^*([\varepsilon^{-1}r'']))\Bigg\} \end{aligned} \tag{4.13}$$

where $r_1 = (r_1, 0)$, $r_2 = (0, r_2)$, $\partial/\partial r_i \mathscr{G}_t(|r|) = (1/\sqrt{t})\mathscr{G}_t^{(i)}(|r|)$ and $\partial^2/\partial r_1 \partial r_2 \mathscr{G}_t(|r|) = (1/t)\, \mathscr{G}_t^{(1,2)}(|r|)$. By Theorem 5.3 (or Theorem 6.1 in 3D) we have that there is c such that

$$P^{\varepsilon}_{\mu^{\varepsilon}}(A^c) \le c|\log \varepsilon|^{2\eta' - 1 + 4\theta} \tag{4.14}$$

a straightforward calculation shows that in the general case the exponent is $2\eta' - d/2 + 2d\theta$. Because $d/2 - 2\eta > d/2 - 2\eta'$, by (4.10) and (4.14), we have that (4.4) is proved by choosing $\bar{u} = d/2 - 2\eta' - 2d\theta$ and by showing that $\bar{u}$ can be chosen strictly positive. The condition $\bar{u} > 0$ is equivalent to $\eta' < d(1/4 - \theta)$: the result follows from $\theta < 1/6$ and by the arbitrary choice of η' ($\eta' \in (\eta, d(1/4 - \theta))$). $\square$

Proposition 4.1 tells us that the solution of the linearized system is uniformly bounded by a quantity vanishing with ε for all times up to the critical one. Now we have to take care of the nonlinearity. With this we mean that we want to prove a similar bound for m_ε (actually we will need and obtain more: we will bound $\| m_\varepsilon - l_\varepsilon \|$). Let us start with a lemma.

Lemma 4.2. *For any $p \in (0, \alpha a)$ and $u > 0$ there is c such that*

$$P^{\varepsilon}_{\mu^{\varepsilon}}(\{\| m_\varepsilon(\cdot, \varepsilon^{1/10}; \delta_{\sigma^*})\| \le \varepsilon^{2\alpha a - p}\}) \ge 1 - c\varepsilon^{u}. \tag{4.15}$$

Proof. First of all we will use the following formula:

$$m_\varepsilon(r, t; \delta_{\sigma^*}) = \int \mathrm{d}r' \mathscr{G}_t(r - r') \mathrm{e}^{\alpha t} \sigma^*([\varepsilon^{-1} r']) - \beta \int_0^t \mathrm{d}s\, \mathrm{e}^{\alpha(t-s)}$$
$$\times \int \mathrm{d}r' \mathscr{G}_{t-s}(r - r') m_\varepsilon^3(r', s; \delta_{\sigma^*}). \tag{4.16}$$

By the monotonic properties of (4.2) we have that $|m_\varepsilon(r, t; \delta_{\sigma^*})| \le 1$ for all t. Hence

$$\| m_\varepsilon(r, \varepsilon^{1/10}; \delta_{\sigma^*}) - l_\varepsilon(r, \varepsilon^{1/10}; \delta_{\sigma^*})\| \le \beta \int_0^{\varepsilon^{1/10}} \mathrm{e}^{-\alpha(s - \varepsilon^{1/10})} \mathrm{d}s \int \mathrm{d}r' \mathscr{G}_{\varepsilon^{1/10} - s}(r - r')$$
$$\times |m_\varepsilon(r', s; \delta_{\sigma^*})| \le \mathrm{e}^{\alpha \varepsilon^{1/10}} \varepsilon^{1/10} \le 2\varepsilon^{1/10}. \tag{4.17}$$

Remembering that $\alpha a < 1/30$, we observe that we are allowed to replace m_ε and l_ε in formula (4.15). In formula (4.15) we want also to get rid of the sup over an uncountable set. To this purpose we observe that there is c such that

$$\left| \frac{\partial l_\varepsilon}{\partial r}(r, \varepsilon^{1/10}; \delta_{\sigma^*}) \right| \le \frac{c}{\varepsilon^{1/20}} \tag{4.18}$$

and so, using (4.17) and (4.18)

$$\boldsymbol{P}_{\mu^\varepsilon}^{\varepsilon}(\{\|m_\varepsilon(\cdot,\varepsilon^{1/10};\delta_{\sigma^*})\| \geq \varepsilon^{2\alpha a-p}\})$$

$$\leq \boldsymbol{P}_{\mu^\varepsilon}^{\varepsilon}\left(\sup_{r\in[0,|\log\varepsilon|^{1/2+\theta}]^d\cap\varepsilon^\zeta \mathbb{Z}} |l_\varepsilon(\boldsymbol{r},\varepsilon^{1/10};\delta_{\sigma^*})| \geq \frac{\varepsilon^{2\alpha a-p}}{2}\right)$$

$$\leq \varepsilon^{-\zeta d}(|\log\varepsilon|^{1/2+\theta})^d \sup_r \boldsymbol{P}_{\mu^\varepsilon}^{\varepsilon}\left(|l_\varepsilon(\boldsymbol{r},\varepsilon^{1/10};\delta_{\sigma^*})| \geq \frac{\varepsilon^{2\alpha a-p}}{2}\right) \tag{4.19}$$

Provided $\zeta > (2\alpha a - p) + 1/20$ (choose $\zeta = 1/4$). By using the Chebychev inequality with moment $2n$ we get that the chain of inequalities in (4.19) can be continued with

$$\varepsilon^{-d/4}(|\log\varepsilon|^{1/2+\theta})^d\varepsilon^{-(2\alpha a-p)2n}2^{2n}\int\cdots\int \mathrm{d}\boldsymbol{r}_1\cdots\boldsymbol{r}_{2n}\prod_{i=1}^{2n}\mathscr{G}_{\varepsilon^{1/10}}(\boldsymbol{r}-\boldsymbol{r}_i)$$

$$\times \boldsymbol{E}_{\mu^\varepsilon}^{\varepsilon}\left(\prod_{i=1}^{2n}\sigma([\varepsilon^{-1}\boldsymbol{r}_i])\right) \tag{4.20}$$

the expectation in this formula is exactly v_{2n} when the points $\boldsymbol{r}_i$ are separated from each other at least by a distance ε (as in (4.9)). Again, if $|\boldsymbol{r}_i - \boldsymbol{r}_j| < \varepsilon$ we have that $\sigma([\varepsilon^{-1}\boldsymbol{r}_i])\sigma([\varepsilon^{-1}\boldsymbol{r}_j]) = 1$ and so we will have to deal with a lower-order correlation function (but we can take advantage of the restriction on the domain of integration). By using Theorem (5.3) or Theorem (6.1) (respectively, in $d = 2$ and $d = 3$) we get that, chosen $b \in (0, p)$, there is c such that

$$\boldsymbol{P}_{\mu^\varepsilon}^{\varepsilon}(\{\|m_\varepsilon(\cdot,\varepsilon^{1/10};\delta_{\sigma^*})\| \geq \varepsilon^{2\alpha a-p}\})$$

$$\leq c\varepsilon^{-d/4}|\log\varepsilon|^{(1/2+\theta)d}\varepsilon^{-(2\alpha a-p)2n}\varepsilon^{2n(\alpha a-b)} \leq c\varepsilon^{-d/3}\varepsilon^{(p-b)2n}. \tag{4.21}$$

by choosing n such that $2n(p-b) - d/3 \geq u$ we get the result. □

Now we have a control over m_ε at a time that is very short and the linearized is under control. These two elements will be combined by means of the following Lemma.

Lemma 4.3. *Fix* $p \in (0, \alpha a/2)$ *and* $\eta > 0$. *If* $\|m_\varepsilon(\cdot,\varepsilon^{1/10};\delta_{\sigma^*})\| \leq \varepsilon^{2\alpha a-p}$ *and if* $\|l_\varepsilon(\cdot,t_a;\delta_{\sigma^*})\| \leq \varepsilon^{\alpha a}/|\log\varepsilon|^\eta$ *we have*

$$\|m_\varepsilon(\cdot,2t_a;\delta_{\sigma^*})\| \leq c|\log\varepsilon|^{-\eta} \tag{4.22}$$

and

$$\|m_\varepsilon(\cdot,2t_a;\delta_{\sigma^*}) - l_\varepsilon(\cdot,2t_a;\delta_{\sigma^*})\| \leq c|\log\varepsilon|^{-3\eta} \tag{4.23}$$

and formulae (4.22) *and* (4.23) *are a fortiori true if we consider* $m_\varepsilon(\cdot,t;\delta_{\sigma^*})$ (*analog for* l) *with* t *smaller than* $2t_a$ (*say* $t \in [t_a, 2t_a]$).

Combining Proposition 4.1 and Lemmas 4.2 and 4.3 we get the following Proposition, whose proof is obvious.

Proposition 4.4. *Given* η *as in Proposition* 4.1, *there exist* $\bar{u} > 0$, $c > 0$ *and* $c' > 0$ *such that*

$$\begin{aligned} P^{\varepsilon}_{\mu^{\varepsilon}}(\{\| m_{\varepsilon}(\cdot, 2t_a; \delta_{\sigma^*})\| \le c|\log\varepsilon|^{-\eta}\} \cap \{\| m_{\varepsilon}(\cdot, 2t_a; \delta_{\sigma^*}) \\ - l_{\varepsilon}(\cdot, 2t_a; \delta_{\sigma^*})\| \le c|\log\varepsilon|^{-3\eta}\}) \ge 1 - c'|\log\varepsilon|^{-\bar{u}} \end{aligned} \tag{4.24}$$

and (4.24) *is a fortiori true if we consider* m *and* l *at time* $t \in [t_a, 2t_a]$.

Proof of Lemma 4.3. By the monotonicity of (4.2) we have

$$|m_{\varepsilon}(\boldsymbol{r}, t; \delta_{\sigma^*})| \le m(t - \varepsilon^{1/10}) \tag{4.25}$$

for $t > \varepsilon^{1/10}$, where $m(t)$ solves

$$\begin{aligned} m'(t) &= \alpha m - \beta m^3, \\ m(0) &= \varepsilon^{2\alpha a - p}. \end{aligned} \tag{4.26}$$

Hence

$$|m_{\varepsilon}(\boldsymbol{r}, t; \delta_{\sigma^*})| \le \mathrm{e}^{\alpha t}\varepsilon^{2\alpha a - p} \tag{4.27}$$

and in particular $|m_{\varepsilon}(\boldsymbol{r}, t; \delta_{\sigma^*})| \le \varepsilon^{\alpha a - b}$. Moreover set $h_{\varepsilon}(t) = \| m_{\varepsilon}(\cdot, t; \delta_{\sigma^*}) - l_{\varepsilon}(\cdot, t; \delta_{\sigma^*})\|$. Using (4.27) and the integral representations of m_{ε} (4.16) and l_{ε} (4.6) we have

$$h_{\varepsilon}(t) \le \mathrm{e}^{\alpha(t - \varepsilon^{1/10})} h_{\varepsilon}(\varepsilon^{1/10}) + c\int_{\varepsilon^{1/10}}^{t} \mathrm{d}s\, \mathrm{e}^{\alpha(t-s)}[\mathrm{e}^{\alpha s}\varepsilon^{2\alpha a - p}]^3 \tag{4.28}$$

and so

$$h_{\varepsilon}(t_a) \le c\varepsilon^{3(\alpha a - p)} \tag{4.29}$$

(observe $h_{\varepsilon}(\varepsilon^{1/10}) \le \beta\int_0^{\varepsilon^{1/10}} \mathrm{d}s\, \mathrm{e}^{\alpha(t-s)}\| m_{\varepsilon}^3 \| \le \beta\varepsilon^{1/10}$ and remember that $3\alpha a < 1/10$). By the hypotheses and (4.29) we have that

$$\| m_{\varepsilon}(\cdot, t_a; \delta_{\sigma^*})\| \le c\frac{\varepsilon^{\alpha a}}{|\log\varepsilon|^{\eta}} \tag{4.30}$$

and so (exactly as in the proof (4.27)) if $t \ge t_a$ we get

$$\| m_{\varepsilon}(\cdot, t; \delta_{\sigma^*})\| \le c\mathrm{e}^{\alpha(t - t_a)}\frac{\varepsilon^{\alpha a}}{|\log\varepsilon|^{\eta}} \tag{4.31}$$

and if we choose $t = 2t_a$ we have (4.22). Furthermore,

$$h_{\varepsilon}(t) \le \mathrm{e}^{\alpha(t - t_a)} h_{\varepsilon}(t_a) + c\int_{t_a}^{t} \mathrm{d}s\, \mathrm{e}^{\alpha(t-s)}\left[\frac{\mathrm{e}^{\alpha s}\varepsilon^{2\alpha a}}{|\log\varepsilon|^{\eta}}\right]^3. \tag{4.32}$$

Set $t = 2t_a$ and use (4.29) for the first term to get

$$h_\varepsilon(t_a) \le \varepsilon^{2\alpha a - 3p} + c\varepsilon^{-2\alpha a}\frac{\varepsilon^{6\alpha a}}{|\log\varepsilon|^{3\eta}}\left(\frac{e^{4\alpha t_a} - e^{2\alpha t_a}}{2\alpha}\right) \le \varepsilon^{2\alpha a - 3p}$$
$$+ \frac{c}{\alpha}\frac{1}{|\log\varepsilon|^{3\eta}} \le \frac{c'}{|\log\varepsilon|^{3\eta}}, \tag{4.33}$$

i.e., (4.23). By (4.31) and (4.32) we see that (4.22) and (4.33) are a fortiori true if $t \in [t_a, 2t_a]$. □

So we now have an upper bound and we are allowed to say that up to $t = t_c$ the escape does not happen (the magnetization is still uniformly bounded by a quantity vanishing with ε). Now we are looking for a lower bound: we give the following two lemmas, the first of which will be crucial to determine the spatial structure of the random field after the escape.

Lemma 4.5. *For all $\zeta > 0$ there is q such that*

$$P^\varepsilon_{\mu^\varepsilon}\left(\left\{|l_\varepsilon(r, 2t_a; \delta_{\sigma^*})| \le \frac{q}{|\log\varepsilon|^{d/4}}\right\}\right) \le \zeta. \tag{4.34}$$

Proof. We have that

$$l_\varepsilon(r, 2t_a; \delta_{\sigma^*}) = e^{2\alpha t_a}\varepsilon^d \sum_x \frac{1}{(4\pi a|\log\varepsilon|/d)^{d/2}} \exp\left\{-\frac{(r - \varepsilon x)^2}{4a|\log\varepsilon|/d}\right\}\sigma^*(x) + R^\varepsilon(r)$$
$$= \frac{e^{-\alpha t^*}\delta^{d/2}}{|\log\varepsilon|^{d/4}} \sum_x \frac{1}{(4\pi a/d)^{d/2}} \exp\left\{-\frac{(r|\log\varepsilon|^{-1/2} - \delta x)^2}{4a/d}\right\}\sigma^*(x) + R^\varepsilon(r) \tag{4.35a}$$

and

$$R^\varepsilon(r) = \int dr'\, e^{2\alpha t_a}\sigma^*([\varepsilon^{-1}r'])\mathscr{G}_{2t_a}(r - r')$$
$$\times\left[1 - \exp\left(\frac{(r - r')^2 - (r - \varepsilon[\varepsilon^{-1}r'])^2}{4a|\log\varepsilon|/d}\right)\right]. \tag{4.35b}$$

So we have that there is c such that

$$\left|l_\varepsilon(r, 2t_a; \delta_{\sigma^*})|\log\varepsilon|^{d/4} - Y^\varepsilon_{t^*}(\phi_r)\right| \le \frac{d\varepsilon}{2a|\log\varepsilon|}\int dr\, e^{2\alpha t_a}\mathscr{G}_{2t_a}(r)|r|$$
$$\le c\frac{\varepsilon^{1 - 2\alpha a}}{\sqrt{|\log\varepsilon|}} < c\varepsilon^{1/2} \tag{4.36}$$

in which $\phi_r(\cdot) = \mathscr{G}_{2a}(r|\log\varepsilon|^{-1/2} - \cdot)$. From Theorem 3.1 formula (4.34) follows. □

We can strengthen this result by taking into account the following lemma.

Lemma 4.6. *For all $\zeta > 0$ and $q > 0$ there is L such that*

$$\boldsymbol{P}^{\varepsilon}_{\mu^{\varepsilon}}\left(\left\{\sup_{|\boldsymbol{r}-\boldsymbol{r}'| \le L\sqrt{|\log\varepsilon|}} |l_{\varepsilon}(\boldsymbol{r}, 2t_a; \delta_{\sigma^*}) - l_{\varepsilon}(\boldsymbol{r}', 2t_a; \delta_{\sigma^*})| > \frac{q}{|\log\varepsilon|^{d/4}}\right\}\right) \le \zeta \tag{4.37}$$

Proof. By Chebyshev inequality we get (in a similar way as in the proof of Proposition 4.1, see (4.11)–(4.13))

$$\boldsymbol{P}^{\varepsilon}_{\mu^{\varepsilon}}\left(\left\{\sup_{|\boldsymbol{r}-\boldsymbol{r}'| \le L\sqrt{|\log\varepsilon|}} |l_{\varepsilon}(\boldsymbol{r}, 2t_a; \delta_{\sigma^*}) - l_{\varepsilon}(\boldsymbol{r}', 2t_a; \delta_{\sigma^*})| > \frac{q}{|\log\varepsilon|^{d/4}}\right\}\right)$$
$$\le q^{-2}|\log\varepsilon|^{d/2}\boldsymbol{E}^{\varepsilon}_{\mu^{\varepsilon}}\left(D^{(d)}_{L\sqrt{|\log\varepsilon|}}(\boldsymbol{r}, l_{\varepsilon}(\cdot, 2t_a; \delta_{\sigma^*}))\right) \tag{4.38}$$

in $d = 2$ the right-hand side of (4.38) is bounded by $q^{-2}|\log\varepsilon|^{d/2}6L\sqrt{|\log\varepsilon|}$.

$$\left\{\sum_{i=1,2}\int_{r_i - L\sqrt{|\log\varepsilon|}}^{r_i + L\sqrt{|\log\varepsilon|}} \mathrm{d}r_i \int\!\!\int \mathrm{d}\boldsymbol{r}'\,\mathrm{d}\boldsymbol{r}''\frac{\mathrm{e}^{4\alpha t}}{2t_a}\mathcal{G}^{(i)}_{2t_a}(\boldsymbol{r}' - \boldsymbol{r}_i)\mathcal{G}^{(i)}_{2t_a}(\boldsymbol{r}'' - \boldsymbol{r}_i)\boldsymbol{E}^{\varepsilon}_{\mu^{\varepsilon}}(\sigma^*([\varepsilon^{-1}\boldsymbol{r}'])\right.$$
$$\times\sigma^*([\varepsilon^{-1}\boldsymbol{r}''])) + 2L\sqrt{|\log\varepsilon|}\int_{r_1 - L\sqrt{|\log\varepsilon|}}^{r_1 + L\sqrt{|\log\varepsilon|}}\mathrm{d}x\int_{r_2 - L\sqrt{|\log\varepsilon|}}^{r_2 + L\sqrt{|\log\varepsilon|}}\mathrm{d}y\int\!\!\int\mathrm{d}\boldsymbol{r}'\,\mathrm{d}\boldsymbol{r}''\frac{\mathrm{e}^{4\alpha t}}{4t_a^2}$$
$$\left.\times\mathcal{G}^{(1,2)}_{2t_a}(\boldsymbol{r}' - (x,y))\mathcal{G}^{(1,2)}_{2t_a}(\boldsymbol{r}'' - (x,y))\boldsymbol{E}^{\varepsilon}_{\mu^{\varepsilon}}(\sigma^*([\varepsilon^{-1}\boldsymbol{r}'])\sigma^*([\varepsilon^{-1}\boldsymbol{r}'']))\right\}$$
$$\le c_1L^2 + c_2L^4 \tag{4.39}$$

in which we used Theorem 5.3. Choose L such that $c_1L^2 + c_2L^4 \le \zeta$ to conclude. The tridimensional case is analogous (use Theorem 6.1). □

Now we are ready for proving Theorem 2.1.

Proof of Theorem 2.1. First of all we remark that (2.5a) is easily proved for $t \le \tau|\log\varepsilon|$ ($\tau < \tau_c$). In fact in the bidimensional case by Theorem 5.3 (choosing b sufficiently small once fixed $\tau < \tau_c$) and in the tridimensional case for $t \le (\tau_c/2)|\log\varepsilon|$ by Theorem 5.3 and otherwise by Theorem 6.3 we have that for all n

$$\lim_{\varepsilon\to 0} v^{\varepsilon}_{2n}(\underline{x}, t) = 0, \qquad t = \tau|\log\varepsilon|, \quad \tau < \tau_c \tag{4.40}$$

uniformly in $\underline{x} \in \mathcal{M}^{\varepsilon}_{2n}$. Recall that $v_{2n+1} \equiv 0$ and so (2.5a) follows for $\tau < \tau_c$. It remains to prove (2.5a) in a neighborhood of $t = t_c$ (for instance $t \in [(\tau_c - a)|\log\varepsilon|, \tau_c|\log\varepsilon|]$, a as in (4.1)) and (2.5b)). To this purpose remember that by Theorem 5.1 we have that for all $n \ge 1$ and for $t \in [0, a^*|\log\varepsilon|]$

$$\lim_{\varepsilon\to 0}\sup_{\underline{x}\in\mathcal{M}^{\varepsilon}_{n}}\left|\mu^{\varepsilon}_t\left(\prod_{i=1}^{n}\sigma(x_i)\right) - \boldsymbol{E}^{\varepsilon}_{\mu^{\varepsilon}}\left(\prod_{i=1}^{n}m_{\varepsilon}(\varepsilon x_i, t - t^*; \delta_{\sigma^*})\right)\right| = 0. \tag{4.41}$$

From the definitions in (4.1) it follows that $t_c - t^*$ and $t_f - t^*$ belong to $[0, a^*|\log \varepsilon|]$. Proposition 4.4 (formula (4.24)) tells us that

$$\lim_{\varepsilon \to 0} \boldsymbol{P}^{\varepsilon}_{\mu^{\varepsilon}}(\{ \| m_\varepsilon(\cdot, t; \delta_{\sigma^*}) \| > c|\log \varepsilon|^{-\eta}\}) = 0, \quad t \in [t_a, 2t_a] \tag{4.42}$$

and so by the fact that $\eta > 0$ and by (4.41) we complete the proof of (2.5a).

Now consider the case $t = t_f$ in (4.41). We fix $\underline{x} \in \mathcal{M}^{\varepsilon}_n$ and define

$$\mathcal{G}_i(\varepsilon) \equiv \left\{ |m_\varepsilon(\boldsymbol{r}, 2t_a; \delta_{\sigma^*})| \geq \frac{q}{|\log \varepsilon|^{d/4}} \quad \text{for all } |\boldsymbol{r} - \varepsilon \boldsymbol{x}_i| \leq L\sqrt{|\log \varepsilon|} \right\} \cap$$
$$\left\{ \| l_\varepsilon(\cdot, 2t_a; \delta_{\sigma^*}) - m_\varepsilon(\cdot, 2t_a; \delta_{\sigma^*}) \| < \frac{c}{|\log \varepsilon|^{3\eta}} \right\}. \tag{4.43}$$

By the choice of η in Proposition 4.4 ($3\eta > d/4$) we have that for an arbitrary $q' > q$ and ε sufficiently small

$$\mathcal{G}_i(\varepsilon) \supset \left\{ |l_\varepsilon(\boldsymbol{r}, 2t_a; \delta_{\sigma^*})| \geq \frac{q}{|\log \varepsilon|^{d/4}} \quad \text{for all } |\boldsymbol{r} - \varepsilon \boldsymbol{x}_i| \leq L\sqrt{|\log \varepsilon|} \right\} \cap$$
$$\left\{ \| l_\varepsilon(\cdot, 2t_a; \delta_{\sigma^*}) - m_\varepsilon(\cdot, 2t_a; \delta_{\sigma^*}) \| < \frac{c}{|\log \varepsilon|^{3\eta}} \right\} \tag{4.44}$$

By Proposition 4.4, Lemmas 4.5 and 4.6 and by (4.44), for every m and $\zeta > 0$ there are L and q (q') such that

$$\lim_{\varepsilon \to 0} \boldsymbol{P}^{\varepsilon}_{\mu^{\varepsilon}}\left(\bigcap_{i=1}^{m} \mathcal{G}_i(\varepsilon) \right) \geq 1 - \zeta. \tag{4.45}$$

Lemma 4.8 of De Masi et al. (1991a) proves that for all $L > 0$

$$\lim_{\varepsilon \to 0} \sup{}^* |m(\boldsymbol{0}, |\log \varepsilon|^{1/3}; \psi) - m(\boldsymbol{0}, |\log \varepsilon|^{1/3}; \psi')| = 0 \tag{4.46}$$

in which $m(\cdot, t; \varphi)$ is the solution of (2.4) with initial datum φ and sup* means that we have to take the supremum over the functions ψ, ψ' such that $\psi(\boldsymbol{r}) = \psi(\boldsymbol{r}')$ for all $|\boldsymbol{r}| \leq L\sqrt{|\log \varepsilon|}$ and such that $\|\psi\|$ and $\|\psi'\|$ are smaller than 1.

This last result means that the behavior of the solution of (2.4) in a fixed point at time $|\log \varepsilon|^{1/3}$ does not depend on the whole initial datum. We will use this property in the following way: choose a vector of points $\underline{x} \in \mathcal{M}^{\varepsilon}_n$. For each point $\boldsymbol{x}_i$ $(i = 1, \ldots, m)$ we can consider a new function $m'_\varepsilon(\boldsymbol{r})$ equal to $m_\varepsilon(\cdot, 2t_a; \delta_{\sigma^*})$ in the region in which $|m_\varepsilon| \geq q/|\log \varepsilon|^{d/4}$ and outside this region $m'_\varepsilon(\boldsymbol{r}) \equiv q/|\log \varepsilon|^{d/4}$. By (4.46) and by the continuity of $m_\varepsilon(\cdot, 2t_a; \delta_{\sigma^*})$ we know that with large probability the region in which m_ε and m'_ε coincide contains a ball of diameter $L\sqrt{|\log \varepsilon|}$. By (4.46)

$$\lim_{\varepsilon \to 0} |m_\varepsilon(\boldsymbol{x}_i, t_f - t_c; m_\varepsilon(\cdot, 2t_a; \delta_{\sigma^*})) - m_\varepsilon(\boldsymbol{x}_i, t_f - t_c; m'_\varepsilon(\cdot))| = 0. \tag{4.47}$$

So we can study $m_\varepsilon(\cdot, t_f; m'_\varepsilon(\cdot))$ in order to recover the behavior of the system in $\boldsymbol{x}_i$. Because $m'_\varepsilon(\boldsymbol{r}) \geq q/|\log \varepsilon|^{d/4}$ for all $\boldsymbol{r}$ and by the monotonicity properties of the RD equation (2.4) we have that

$$m_\varepsilon(\boldsymbol{r}, t_f - t_c; m'_\varepsilon(\cdot)) \geq m(|\log \varepsilon|^{1/3}) \tag{4.48}$$

and $m(t)$ is the solution of

$$\dot{m} = \alpha m - \beta m^3,$$
$$m(0) = q/|\log \varepsilon|^{d/4}.$$

We obtain that

$$m(|\log \varepsilon|^{1/3}) = \frac{\exp(\alpha |\log \varepsilon|^{1/3})}{\sqrt{|\log \varepsilon|^{d/2} q^{-2} + (\exp(2\alpha |\log \varepsilon|^{1/3}) - 1)(\beta/\alpha)}} \to \sqrt{\frac{\alpha}{\beta}} = m^* \tag{4.49}$$

(the limit is for $\varepsilon \to 0$) but for $\sigma(\cdot, t^*) \in \bigcap_{i=1}^m \mathscr{G}_i$ we have that

$$|m_\varepsilon(\varepsilon \boldsymbol{x}_i, 2t_a; \delta_{\sigma^*}) - \operatorname{sign}(l_\varepsilon(\varepsilon \boldsymbol{x}_i, 2t_a; \delta_{\sigma^*}))|m_\varepsilon(\varepsilon \boldsymbol{x}_i, 2t_a; \delta_{\sigma^*})|| \leq \frac{c}{|\log \varepsilon|^{3\eta'}}. \tag{4.50}$$

Hence by (4.47)–(4.50) we obtain that for $\sigma(\cdot, t^*) \in \bigcap_{i=1}^m \mathscr{G}_i$

$$\lim_{\varepsilon \to 0} m_\varepsilon(\varepsilon \boldsymbol{x}_i, t_f - t^*; \delta_{\sigma^*}) \geq m^* \operatorname{sign}(l_\varepsilon(\varepsilon \boldsymbol{x}_i, 2t_a; \delta_{\sigma^*})). \tag{4.51}$$

On the other side by the maximum principle $\|m_\varepsilon\| \leq 1$ and so by repeating steps (4.48) and (4.49) with $m(0) = 1$ we have that formula (4.51) holds as an equality. By (4.36) we already know that the sign of l_ε can be approximated by the sign of a gaussian random field with zero mean and the right covariance. By using (4.45) we end the proof. □

5. Estimates on correlation functions I: General Theorems and the bidimensional case

General remark. Practically in all the proofs in Chapters 5–7 a constant c (also c' and c'') will appear many times without being defined. We will do this every time the meaning of this constant is implicitly clear.

We start with some definitions. For any $n \geq 1$ and for any function f on $\mathscr{M}_n^\varepsilon$ let

$$\begin{aligned} Lf(\underline{x}) = \frac{1}{2d} \sum_{l=1}^{d} \sum_{i=1}^{n} \sum_{b=\pm 1} \Big\{ & \mathbf{1}_{(x_i + be_l \neq x_j \, \forall j \neq i)} [f(\underline{x}(i), \boldsymbol{x}_i + be_l) - f(\underline{x})] \\ & + \sum_{k=1}^{n} \mathbf{1}_{(x_k = x_i + be_l)} [f(\underline{x}(i,k), \boldsymbol{x}_k, \boldsymbol{x}_i) - f(\underline{x})] \Big\} \end{aligned} \tag{5.1}$$

in which $\boldsymbol{e}_l$ is the unit vector in the direction l, $\underline{x}(i) \in \mathscr{M}_{n-1}^\varepsilon$ $(i = 1, \ldots, n)$ is the configuration $\underline{x}$ without the ith particle and $\underline{x}(i,j) \in \mathscr{M}_{n-2}^\varepsilon$ ($i \neq j$ and belongs to $\{1, \ldots, n\}$) is the configuration $\underline{x}$ without the ith and the jth particles.

Observation. By the expression $(\underline{x}(i), x)$ we mean that the particle with label i moves to x. Obviously this expression is meaningful only if $x \notin \underline{x}(i)$. In general the particles must be moved following the order of the new position, i.e. $(\underline{x}(i,k), x_k, x_i)$ means that the new configuration is obtained from the preceding by exchanging the position of particles i and k. Furthermore given $\underline{x} \in \mathscr{M}_n^\varepsilon$, by $(\underline{x}, x)$ we mean a new configuration belonging to $\mathscr{M}_{n+1}^\varepsilon$ and the particle $n+1$ is at x. There are obvious combinations of the preceding definitions. The process generated by (5.1) is called *stirring* process. We will call $\boldsymbol{P}_t^\varepsilon(\underline{x} \to \underline{y})$ the kernel of the process generated by $(\varepsilon^{-2} + \gamma)L(\gamma$ enters in (2.3a)) and we will indicate with $\boldsymbol{E}_{\underline{x}}^\varepsilon$ its expectation (if the initial configuration is $\underline{x}$). Furthermore we set

$$\| v_n(t) \| \equiv \sup_{\underline{x} \in \mathscr{M}_n^\varepsilon} |v_n^\varepsilon(\underline{x}, t)|. \tag{5.2}$$

Theorem 5.1 (Short time estimate). *In every fixed dimension there are a^*, δ^*, β^* strictly positive such that for any ε and for any product measure λ on $(X_\varepsilon^d, \mathscr{B}(X_\varepsilon^d))$ there is a sequence $\{c_n\}_{n \geq 1}$ such that*

$$\sup_{\varepsilon^{\beta^*} \leq t \leq t_{a^*}} \sup_{\underline{x} \in \mathscr{M}_n^\varepsilon} |v_n^\varepsilon(\underline{x}, t : \lambda)| \leq c_n \varepsilon^{\delta^* n}, \quad t_{a^*} \equiv a^* |\log \varepsilon|. \tag{5.3}$$

Proof. See Chapters IX and X of De Masi and Presutti (1991b), Theorem 9.2.1. The proof reported there is done for a unidimensional system in a smaller space. Nevertheless the proof works also in higher dimension and in larger spaces with trivial modifications. □

Remark. A particular case of λ is that in which $\lambda = \delta_\sigma$, where $\sigma \in X_\varepsilon^d$.

Theorem 5.2. (Avoiding the problem of divergences). *For all $d \geq 2$, for all positive $\tau < T_c$ and for all $b > 0$ there is a sequence $\{c_n\}_{n \geq 1}$ such that*

$$\| v_{2n}(t) \| \leq c_n e^{2n\alpha t} \varepsilon^{2n(1-b)} \tag{5.4}$$

for $t \in [0, \tau |\log \varepsilon|]$.

Proof. This problem is contained in the proof of the Theorem 2.3 of De Masi et al. (1991a). It is sufficient to repeat the steps in Section 5 of De Masi et al. (1991a) taking into account that in $d \geq 2$, given b' positive, there is c such that

$$\sup_{\underline{x} \in \mathscr{M}_{2n}^\varepsilon} \sum_{\underline{y} \in \mathscr{M}_{2n}^\varepsilon} \boldsymbol{P}_t^\varepsilon(\underline{x} \to \underline{y}) \mathbf{1}_{(y_i - y_j = \boldsymbol{q})} \leq \frac{c}{(\varepsilon^{-2}(t) + 1)^{1-b'}} \tag{5.5}$$

in which $\boldsymbol{q} \in \mathbb{Z}^d$ and $i \neq j$. □

Remark 1. In (5.4) we can choose b as small as we want. Many times we will use (5.4), this fact will be implicit.

Remark 2. We define the time of escape as $\tau_c(d) = d/2\alpha$, with d the dimension of the space. In this section we will set $\tau_c = \tau_c(2)$ (and in the next one $\tau_c = \tau_c(3)$).

Remark 3. (5.5) is far from being optimal for large t (see (5.9) below). It has a better behavior for small t. We will use many times (5.5). In order to prove Theorem 5.2 choose $b' \in (0, b)$.

Theorem 5.3 (Uniform estimate on v_2 in the 2D case). *For all $\tau_1 \in (0, \tau_c)$, $\tau_2 \in (\tau_1, \tau_c)$ there is c such that*

$$\| v_2(t) \| \le \frac{c e^{2\alpha t} \varepsilon^d}{|\log \varepsilon|^{d/2}} \quad (d = 2) \tag{5.6}$$

for all $t \in (\tau_1 |\log \varepsilon|, \tau_2 |\log \varepsilon|)$.

Proof. By Lemma 7.1 and the observation that $|v_{2n}^\varepsilon| \le 1$, there is c' such that

$$|v_2^\varepsilon(\underline{x}, t)| \le c' \sup_{i,j} \int_0^t \mathrm{d}s\, e^{2\alpha(t-s)} \{ \boldsymbol{E}_{\underline{x}}^\varepsilon(\mathbf{1}_{(|x_i(t-s) - x_j(t-s)| \le 2)}) + \| v_4^\varepsilon(s) \| \}. \tag{5.7}$$

We split the integral containing the expectation into two pieces ($\int_0^T + \int_T^t$ with $T \in (0, t)$) and for the term containing the norm of v_4^ε we use Theorem 5.2. Hence

$$\begin{aligned} |v_2^\varepsilon(\underline{x}, t)| \le c' \sup_{i,j} \Bigg\{ & \int_0^T \boldsymbol{E}_{\underline{x}}^\varepsilon(\mathbf{1}_{(|x_i(t-s) - x_j(t-s)| \le 2)}) \mathrm{d}s\, e^{2\alpha(t-s)} \\ & + \int_T^t \boldsymbol{E}_{\underline{x}}^\varepsilon(\mathbf{1}_{(|x_i(t-s) - x_j(t-s)| \le 2)}) \mathrm{d}s\, e^{2\alpha(t-s)} \\ & + (e^{2\alpha t} \varepsilon^2) \cdot \left(\varepsilon^{2-4b} \frac{c_4}{2\alpha} e^{2\alpha t} \right) \Bigg\}. \end{aligned} \tag{5.8}$$

By using

$$\sup_{\underline{x} \in \mathcal{M}_{2n}^\varepsilon} \sum_{\underline{y} \in \mathcal{M}_{2n}^\varepsilon} \boldsymbol{P}_{t-s}^\varepsilon(\underline{x} \to \underline{y}) \mathbf{1}_{(y_i - y_j = q)} \le \frac{c}{(\varepsilon^{-2}(t-s) + 1)^{d/2}} \tag{5.9}$$

($\boldsymbol{q} \in \mathbb{Z}^d$) in the first integral (with $d = 2$) and (5.5) in the second one, we obtain

$$\begin{aligned} |v_2^\varepsilon(\underline{x}, t)| \le c' \Bigg\{ & c \frac{\varepsilon^2 e^{2\alpha t}}{(t-T)} \int_0^T e^{-2\alpha s} \mathrm{d}s + \frac{c}{b} (\varepsilon^{2-2b} e^{-2\alpha T} t^{2b}) e^{2\alpha t} \\ & + \frac{c_4}{2\alpha} (e^{2\alpha t} \varepsilon^2)(e^{2\alpha t} \varepsilon^{2-4b}) \Bigg\}. \end{aligned} \tag{5.10}$$

Now choose $T = (3b/2\alpha)|\log\varepsilon| > (2b/2\alpha)|\log\varepsilon|$ and b small enough to obtain $\zeta = -4b + 2\alpha(\tau_c - \tau_2) > 0$ and $\tau_1 \geq 3b/\alpha$. With these choices there are c and c' (possibly different from the previous constants c and c') such that

$$|v_2(\underline{x}, t)| \leq c\frac{\varepsilon^2 e^{2\alpha t}}{|\log\varepsilon|}(1 + c'\varepsilon^{\zeta'}) \tag{5.11}$$

provided $\zeta' \in (0, \zeta)$ and this proves the Theorem. $\square$

What is missing in this theorem is an estimate at short times. Actually the estimate (5.6) really does not work for short times, but something weaker holds.

Theorem 5.4 (Nonuniform estimate on v_2 in the 2D case). *Fix* $\tau \in (0, \tau_c)$. *For all* $\phi \in \mathscr{S}(\mathbb{R}^2)$ *there is* c *such that*

$$e^{-2\alpha t}\delta^d \sum_{x \neq y} \phi(\delta x)\phi(\delta y)v_2(x, y; t) \leq c \tag{5.12}$$

for all $t \leq \tau|\log\varepsilon|$ (*remember that* $\delta = \varepsilon/\sqrt{|\log\varepsilon|}$ *and* $d = 2$).

Proof. With the same consideration as for (5.7) we have that the left-hand side of (5.12) is bounded by

$$e^{-2\alpha t}\delta^d \sum_{x \neq y} |\phi(\delta x)\phi(\delta y)| \int_0^t e^{-2\alpha s}\, ds [\boldsymbol{E}^{\varepsilon}_{x,y}\{\mathbf{1}_{(|x(t-s)-y(t-s)| \leq 2)}\} \\ + \|v_4(\cdot, s)\|] \equiv I_1 + I_2 \tag{5.13}$$

with obvious meaning of I_1 and I_2. As in the preceding proof we have to consider separately the two terms on the right-hand side of (5.13). For the first one, as in Ravishankar (1989, 1992), we have that

$$I_1 \leq \sum_{x \neq y} |\phi(\delta x)\phi(\delta y)| \int_0^t e^{2\alpha(t-s)}\, ds \sum_{x', y'} \boldsymbol{P}^{\varepsilon}_{t-s}((\boldsymbol{x}, \boldsymbol{y}) \to (\boldsymbol{x}', \boldsymbol{y}'))\mathbf{1}_{(|x'-y'| \leq 2)}$$

$$\leq \delta^d \|\phi(\cdot)\| \sum_y |\phi(\delta y)| \int_0^t e^{-2\alpha}\, ds \sum_z \sum_{e: |e| \leq 2, e \in \mathbb{Z}^d} \boldsymbol{P}^{\varepsilon}_{t-s}((z, z+e) \to (\mathbb{Z}^d, y)), \tag{5.14}$$

where we have used the reversibility of the simple exclusion process and we have summed over $\boldsymbol{x}$. By using again reversibility and the fact that

$$\boldsymbol{P}^{\varepsilon}_t((z, z+\boldsymbol{e}) \to (\mathbb{Z}^d, \boldsymbol{y})) = \boldsymbol{P}^0_t(z \to y) + \boldsymbol{P}^0_t(z + \boldsymbol{e} \to \boldsymbol{y}). \tag{5.15}$$

($\boldsymbol{P}^0_t(\cdot \to \cdot)$ is the kernel of a single particle executing random walk) we have that there is c_d depending only on d such that

$$I_1 \leq c_d \|\phi(\cdot)\| \left(\delta^d \sum_y |\phi(\delta y)|\right)\left[\frac{1 - e^{-2\alpha t}}{2\alpha}\right]. \tag{5.16}$$

So there is a constant $c(\phi)$ that bounds I_1 for all δ (sufficiently small) and $t \geq 0$. *This part of the proof works in every dimension.* Consider the term I_2 in (5.13). By using Theorem 5.2 we have

$$I_2 \leq c\left(\delta^4 \sum_{x \neq y} \phi(\delta x)\phi(\delta y)\right)\delta^{-2}\left(\int_0^t \mathrm{e}^{-2\alpha s}\,\mathrm{d}s\right)\varepsilon^{4-4b}. \tag{5.17}$$

Hence

$$I_2 \leq c(\phi)(\delta^{-2}\varepsilon^2|\log\varepsilon|^{-1})\varepsilon^{2-4b}|\log\varepsilon| \tag{5.18}$$

and choosing b in such a way that $2 - 4b > 0$ we prove the theorem. □

6. Estimates on correlation functions II (the 3D case)

In three dimension the estimate given by Theorem 5.2 is still valid, but we will see that it is not good at all. In order to understand how to solve the problem, we can think of using Lemma 7.1, even in the simpler case ($n = 2$). We can see (See the proofs of Theorems 5.3 and 5.4) that we are able to estimate the term that does not contain any v-function, but we cannot estimate the term containing v_4, without an a priori estimate of $|v_4|$. Such an estimate is given by Theorem 5.2: we do not need an optimal estimation of this term, all the same we need $|v_4|$ at least to be smaller than ε to a certain positive power and this is true only for $t \leq (1 - b)/\alpha)|\log\varepsilon|$ (see formula (5.4)). Clearly a necessary condition for this to happen is to be at a sufficiently short time. So, by using the a *priori estimate* (Theorem 5.2) we will prove the desired estimate at a certain time $\tilde{\tau}|\log\varepsilon|$. We will extend this result to longer times by using 7.2 (that makes use of Theorem 5.2) and in so doing we can find a good lower bound for the escape time.

Theorem 6.1 (Uniform estimates on v_2, the 3D case). *For all* $\tau_1 \in (\tau_c/3, \tau_c)$, $\tau_2 \in (\tau_1, \tau_c)$ *there is a constant such that* (5.6) *holds with* $d = 3$ *for all* $t \in (\tau_1|\log\varepsilon|, \tau_2|\log\varepsilon|)$.

Proof. Choose $\tau \in (\tau_c/3, \tau_c/2)$ (so $\tau_c - 2\tau > 0$ and $\tau - \tau_c/3 > 0$) with the restriction $\tau \leq \tau_1$ and put $t = \tau|\log\varepsilon|$. By applying Lemma 7.1 more than once we obtain

$$|v_2^\varepsilon(\underline{x}, t)| \leq A_1 + A_2 + A_3 + A_4 \tag{6.1}$$

and the A_i, together with their estimates (the sums over the indices (i, j, i', j') will be implicit) are given below.

Let us consider the first

$$A_1 = c\mathrm{e}^{2\alpha t}\int_0^t \mathrm{d}s_1\,\mathrm{e}^{2\alpha s_1}\int_0^{s_1}\mathrm{d}s_2\,\mathrm{e}^{-4\alpha s_2}\|v_6(s_2)\| \leq c'\mathrm{e}^{2\alpha t}\varepsilon^3(\mathrm{e}^{4\alpha t}\varepsilon^{3-6b}) \leq c''\mathrm{e}^{2\alpha t}\varepsilon^3\varepsilon^\zeta, \tag{6.2}$$

where $b = \min(2\alpha((\tau_c - 2\tau)/7, \alpha(\tau - \tau_c/3)/2, 1/10)$ and $\zeta \in (0, 2\alpha(\tau_c - 2\tau) - 6b)$. Formula (6.2) follows from Theorem 5.2 and from the fact that $e^{4\alpha t}\varepsilon^{3-6b} = \varepsilon^{2\alpha(\tau_c - 2\tau) - 6b}$.

$$A_2 = c e^{2\alpha t}\int_0^t e^{2\alpha s_1}\,ds_1 \int_0^{s_1} \frac{e^{-2\alpha s_2}\varepsilon^{2-2b}}{(s_1 - s_2)^{1-b}}\,ds_2 \int_0^{s_2} e^{-2\alpha s_3}\|v_4(s_3)\|\,ds_3$$
$$\le c'\varepsilon^3(e^{4\alpha t}\varepsilon^{3-6b}) \tag{6.3}$$

that is bounded by $c''\varepsilon^{3+\zeta}$ and ζ as in (6.2).

$$A_3 = c e^{2\alpha t}\int_0^t e^{2\alpha s_1}\,ds_1 \int_0^{s_1} e^{-2\alpha s_2}\,ds_2 \int_0^{s_2} ds_3\, e^{-2\alpha s_3}$$
$$\times \boldsymbol{E}^{\varepsilon}_{\underline{x}}\{\mathbf{1}_{(|x_i(t-s_2) - x_j(t-s_2)| \le 2)}\mathbf{1}_{(|x_{i'}(t-s_3) - x_{j'}(t-s_3)| \le 2)}\}. \tag{6.4}$$

Now we separate the integrals into two pieces, starting from the one in s_1.

$$A_3 \le c e^{2\alpha t}\int_0^T e^{2\alpha s_1}\,ds_1 \int_0^{s_1} e^{-2\alpha s_2}\,ds_2 \int_0^{s_2} \boldsymbol{E}^{\varepsilon}_{\underline{x}}\{\mathbf{1}_{(|x_i(t-s_2) - x_j(t-s_2)| \le 2)}\}$$
$$\times \frac{\varepsilon^{2-2b}}{(s_2 - s_3)^{1-b}} e^{-2\alpha s_3}\,ds_3 + c e^{2\alpha t}\int_T^t e^{2\alpha s_1}\,ds_1 \left(\int_0^T + \int_T^{s_1}\right) ds_2\, e^{-2\alpha s_2}\int_0^{s_2} ds_3\, e^{-2\alpha s_3}$$
$$\times \boldsymbol{E}^{\varepsilon}_{\underline{x}}\{\mathbf{1}_{(|x_i(t-s_2) - x_j(t-s_2)| \le 2)}\mathbf{1}_{(|x_{i'}(t-s_3) - x_{j'}(t-s_3)| \le 2)}\}$$

in which $T \in (0, t)$ and we used (5.6) after noticing that

$$\boldsymbol{E}^{\varepsilon}_{\underline{x}}\{\mathbf{1}_{(|x_i(t-s_2) - x_j(t-s_2)| \le 2)}\mathbf{1}_{(|x_{i'}(t-s_3) - x_{j'}(t-s_3)| \le 2)}\}$$
$$= \boldsymbol{E}^{\varepsilon}_{\underline{x}}\{\mathbf{1}_{(|x_i(t-s_2) - x_j(t-s_2)| \le 2)}\boldsymbol{E}^{\varepsilon}_{\underline{x}}\{\mathbf{1}_{(|x_{i'}(t-s_3) - x_{j'}(t-s_3)| \le 2)}|\mathscr{F}_{t-s_2}\}\}$$
$$\le \boldsymbol{E}^{\varepsilon}_{\underline{x}}\{\mathbf{1}_{(|x_i(t-s_2) - x_j(t-s_2)| \le 2)}\}\sup_{\underline{y}} \boldsymbol{E}^{\varepsilon}_{\underline{y}}\{\mathbf{1}_{(|y_i(s_2-s_3) - y_j(s_2-s_3)| \le 2)}\} \tag{6.5}$$

in which $\mathscr{F}_t$ is the σ-algebra of the events at time t. Let us go on with estimations

$$A_3 \le c e^{2\alpha t} e^{2\alpha T}\varepsilon^{5-2b}\int_0^T ds_1 \int_0^{s_1} ds_2 \frac{s_2^b}{(t - s_2)^{3/2}} + c e^{2\alpha t}\int_T^t e^{2\alpha s_1}\,ds_1$$
$$\times \int_0^T \frac{\varepsilon^3}{(s_1 - s_2)^{3/2}}\,ds_2 \int_0^{s_2} \frac{\varepsilon^{2-2b}}{(s_2 - s_3)^{1-b}}\,ds_2 + c e^{2\alpha t}\int_T^t e^{2\alpha s_1}\,ds_1$$
$$\times \int_T^{s_1} \frac{e^{-2\alpha T}\varepsilon^{2-2b}}{(s_1 - s_2)^{1-b}}\left\{\int_0^T ds_3 \frac{e^{-2\alpha s_3}\varepsilon^3}{(s_2 - s_3)^{3/2}} + \int_T^{s_2} ds_3 \frac{e^{-2\alpha T}\varepsilon^{2-2b}}{(s_2 - s_3)^{1-b}}\right\} \tag{6.6}$$

in which we used (5.5) (after conditioning as in (6.5) and (5.9), according to the case).

Now choose $T = t/2$ (remember that $t = \tau|\log\varepsilon|$ and $\tau < \tau_c/2$). Hence

$$A_3 \le c e^{2\alpha t}\varepsilon^3\{e^{2\alpha T}\varepsilon^{2-2b}|\log\varepsilon|^{b+1/2} + e^{2\alpha t}\varepsilon^{2-2b}|\log\varepsilon|^{b+1/2}$$
$$+ e^{2\alpha(t-T)}\varepsilon^{2-2b}|\log\varepsilon|^{b+1/2} + e^{2\alpha(t-2T)}\varepsilon^{1-4b}\} \le c e^{2\alpha t}\varepsilon^{3+\zeta} \tag{6.7}$$

for some $\zeta > 0$ (by the previous choice of b).

And the last one

$$A_4 = c\mathrm{e}^{2\alpha t}\int_0^t \mathrm{d}s\,\mathrm{e}^{-2\alpha s}\boldsymbol{E}_{\underline{x}}^{\varepsilon}\{\mathbf{1}_{(x_i(t-s)-x_j(t-s)\le 2)}\} \le c'\mathrm{e}^{2\alpha t}\varepsilon^3\int_0^T \mathrm{d}s\frac{\mathrm{e}^{-2\alpha s}}{(t-s)^{3/2}}$$

$$+\, c'\mathrm{e}^{2\alpha(t-T)}\int_T^t \mathrm{d}s\frac{\varepsilon^{2-2b}}{(t-s)^{1-b}} \le c''\left\{\frac{\mathrm{e}^{2\alpha t}\varepsilon^3}{|\log\varepsilon|^{3/2}} + \mathrm{e}^{2\alpha t}\varepsilon^3(\mathrm{e}^{-2\alpha T}\varepsilon^{-1-2b}\right\} \tag{6.8}$$

by choosing $T = [\tau_c/3 + (\tau - \tau_c/3)/2]|\log\varepsilon|$ we obtain that $\mathrm{e}^{-2\alpha T}\varepsilon^{-1-2b} = \varepsilon^{2(\alpha(\tau-\tau_c/3)-b)}$ and so by the choice of b

$$A_4 \le c\left[\frac{\mathrm{e}^{2\alpha t}\varepsilon^3}{|\log\varepsilon|^{3/2}} + \mathrm{e}^{2\alpha t}\varepsilon^3\varepsilon^{\zeta'}\right] \le c'\frac{\mathrm{e}^{2\alpha t}\varepsilon^3}{|\log\varepsilon|^{3/2}}. \tag{6.9}$$

Collecting all the estimates we have that there is c such that

$$\|v_2(\cdot;t)\| \le c\frac{\varepsilon^3\mathrm{e}^{2\alpha t}}{|\log\varepsilon|^{3/2}} \tag{6.10}$$

for a time $t = \tau|\log\varepsilon|$ and $\tau \in (\tau_c/3, \tau_c/2)$.

Now we choose $a' \in (0, a)$ (a is given by Lemma 7.2) in such a way that there is an integer k such that $\tau + ka' = \tau_2$. By applying Lemma 7.2 k times, with a' as time grid, we conclude the proof. □

Theorem 6.2 (Nonuniform estimates on v_2). *Fix $\tau < \tau_c$. For all $\phi \in \mathscr{S}(\mathbb{R}^3)$ there is c such that*

$$\mathrm{e}^{-2\alpha t}\delta^3\sum_{x\ne y}\phi(\delta x)\phi(\delta y)v_2(x,y;t) \le c \tag{6.11}$$

for all $t \le \tau|\log\varepsilon|$.

Proof. It is clear that this theorem is a trivial consequence of Theorem 6.1 if $t \ge (b' + \tau_c/3)|\log\varepsilon|$, for any $b' > 0$ (choose $b' = \tau_c/20$). For the case $t < (b' + \tau_c/3)|\log\varepsilon|$ we apply more than once Lemma 7.1, as in Theorem 6.1. We already proved (see the first part of the proof of Theorem 5.3) that the term corresponding to A_4 (6.8) is bounded by a constant c_1. We are left with the contributions of the terms A_1 (6.2), A_2 (6.3) and A_3 (6.4). Simply by making the rough estimate (5.6) on every characteristic function in the A_1 and A_2 and using Theorem 5.2 for $\|v_{2n}^{\varepsilon}\|$ we obtain that the left-hand side of (6.11) is bounded by

$$c\mathrm{e}^{4\alpha t}\varepsilon^{3-6b}|\log\varepsilon|^{3/2} + \mathrm{e}^{-2\alpha t}\delta^3\sum_{x\ne y\in\mathbb{R}^3}|\phi(\delta x)\phi(\delta y)|A_3 + c_1, \tag{6.12}$$

where the first term is given by A_1 and A_2, the second one, of course, by A_3 and the last one by A_4. The first term is bounded by ε^{ζ} and $\zeta = 1 - 3/10 - 6b \ge 1/10$. We are left with the term containing A_3. This term is bounded by

$$c\delta^3 \sum_{x \neq y \in \mathbb{R}^3} |\phi(\delta x)\phi(\delta y)| \int_0^t e^{2\alpha s_1} ds_1 \int_0^{s_1} e^{-2\alpha s_2} ds_2$$

$$\times E^{\varepsilon}_{\mu^{\varepsilon}}\{\mathbf{1}_{(|x_i(t-s_2)-x_j(t-s_2)| \leq 2)}\} \int_0^{s_2} \frac{e^{-2\alpha s_3} ds_3}{(s_2 - s_3)^{1-b}} \varepsilon^{2-2b} \tag{6.13}$$

in which we used (5.5). With the same passages as in (5.13)–(5.16) we find out that (6.13) can be bounded by

$$c\|\phi(\cdot)\|\varepsilon^{2-2b}\left(\delta^3 \sum_x |\phi(\delta x)|\right)\left(\int_0^t e^{2\alpha s_1} ds_1 \int_0^{s_1} e^{-2\alpha s_2} ds_2 \int_0^{s_2} \frac{e^{-2\alpha s_3}}{(s_2 - s_3)^{1-b}}\right)$$

$$\leq c(\phi) e^{2\alpha t} \varepsilon^{2-3b} \leq c(\phi)\varepsilon^{1/2} \tag{6.14}$$

by the previous choice of b. □

In the proof of Theorem 3.1 and Lemma 4.1 we need also a rough bound on higher-order correlation functions. Here is a result that fits our necessities.

Theorem 6.3 (Long time behavior of v-functions in 3D). *For all $\xi > 0$ and $\bar{\tau} \in (\tau_c/3, 2\tau_c/3)$ there is a sequence $\{c_n\}_{n>1}$ such that*

$$\sup_{\underline{x} \in \mathcal{M}^{\varepsilon}_{2n}} |v^{\varepsilon}_{2n}(\underline{x}, t)| \leq c_n \varepsilon^{(3-\xi)n} e^{2\alpha n t} \tag{6.15}$$

for all $t \geq \bar{\tau}|\log \varepsilon|$.

Proof. It is clear that we need to prove (6.15) only for $t = \bar{\tau}|\log\varepsilon|$. The whole statement will follow from Lemma 7.2 and (if t is very large) from the fact that we know that $\|v^{\varepsilon}_{2n}\| \leq 1$ for all t. Once again the underlying idea of this proof is simply to iterate the estimate (7.2), paying attention that it is not enough to estimate $\mathbf{1}_{(|x(t-s_j)-y(t-s_j)| \leq 2)}$ by conditioning at time $t - s_{j-1}$ (and making the sup over all the configurations at time $t - s_{j-1}$, as in (6.5)). In fact in this way, in order to avoid the divergence of the time integral $\int_0^t \varepsilon^3 (t-s)^{-3/2} ds$ we should choose an exponent smaller than one (see (5.5)) and clearly this is far from the optimal estimate (we would have ε^{2-b} instead of ε^3 as small factor).

We start by considering Lemma 7.1 that gives a bound for v_{2n} in terms of $v^{\varepsilon}_{2n-2}, v^{\varepsilon}_{2n}, v^{\varepsilon}_{2n+2}$. We iterate it M ($>n$) times ($M(n)$ will be chosen in (6.26)). After this procedure we will have a bound for v^{ε}_{2n} depending on v^{ε}_{2m}, $m = 1, \ldots, n+M$ and the bounding term will have a tree structure, naturally arising from the iteration. This structure will be described by the notation we are going to introduce.

Consider the sequences $\{a^q_i\}_{1 \leq i \leq q}$, with $1 \leq q \leq M$ and $a^q_i \in \{-1, 0, +1\}$ for all $i \in \{1, \ldots, q\}$. We say that $a^q \equiv \{a^q_i\}$ is a path of iteration if either

(i) for $q \in \{1, \dots, M-1\}$

$$n + \sum_{i=1}^{p} a_i^p > 0, \quad \forall p < q \tag{6.16}$$

and equality holds if $p = q$, or

(ii) $q = M$ and (6.16) holds for all $p < M$.

We shall denote by $\mathscr{I}$ the set of all the paths of iteration and by $\mathscr{I}^p$ the set of paths of length p (obviously $p \in \{1, \dots, M\}$, all the $\mathscr{I}^p$ are disjoint and their union is the whole $\mathscr{I}$. When we give a path we select the branches that contain q integrals and such that in the ith integral they contain a characteristic function that asks for two particles to be close (if $a_i^q = 0$ or -1) or a characteristic function that requires that a particle does not have any neighbor (if $a_i^q = +1$). But this is not enough. If $a_i^q = -1$ the two neighbor particles (in the characteristic function) disappear. On the contrary, if a_i^q is equal to $+1$ two particles arise near the particle without neighbors. So, to specify the branch, we have to specify all the labels of the particles which disappear (or arise). In order to do this, we define by induction a family of sets (of labels). For all a^q we give $A_0 = \{1, \dots, 2n\}$. Given A_{k-1} ($k \le q$) we define

$$A_k = \begin{cases} A_{k-1} \cup \{i_k, k_k\} & i_k = \max(A_{k-1}) + 1, k_k = i_k + 1, \quad \text{if } a_k^q = +1, \\ A_{k-1}/\{i_k, k_k\} & i_k, k_k \in A_{k-1}, \quad \text{if } a_k^q = 0, -1. \end{cases} \tag{6.17}$$

The choice of $\{i_k, k_k\}$ in the second step is arbitrary. From this definition a finite sequence of sets $\{A_k\}$ and a vector $\underline{v} \equiv (i_1, \dots, i_q, k_1, \dots, k_q)$ arise. The set of all $\underline{v}$ (given a^q) will be called $\mathscr{L}_{a^q}$. By specifying sequentially q, a^q and $\underline{v}$ we give a branch of the iteration tree. We give also the following definitions ($p \le q$)

$$\chi^{a^q, \underline{v}}(p) = \begin{cases} \mathbf{1}, & \text{if } a_p^q = +1, \\ \mathbf{1}_{(|x_{i_p} - x_{k_p}| \le M+1)}, & \text{if } a_p = 0, -1. \end{cases} \tag{6.18}$$

and

$$m(a^q) = n + \sum_{i=1}^{q} a_i^q. \tag{6.19}$$

We have that

$$\begin{aligned} |v_{2n}^\varepsilon(\underline{x}, t) \le c e^{2\alpha n t} \sum_{q=n}^{M} \sum_{a^q \in \mathscr{I}^q} \sum_{\underline{v} \in \mathscr{L}_{a^q}} \int_0^t e^{2\alpha a_1^q s_1} \dots \int_0^{s_{q-2}} e^{2\alpha a_{q-1}^q s_{q-1}} \, ds_{q-1} \\ \times \int_0^{s_{q-1}} ds_q \, \mathscr{E}_{\underline{x}} \left\{ \prod_{p=1}^{q} \chi^{a^q \underline{v}}(p) \right\} e^{-2\alpha m(a^q) s_q} \| v_{2m(a^q)}(s_q) \|, \end{aligned} \tag{6.20}$$

where $\mathscr{E}_{\underline{x}}$ is a new (branching) process starting at $\underline{x}$. This is simply the stirring process in each interval of time $(t - s_j, t - s_{j+1})$. At time $t - s_{j+1}$ ($s_0 = t$) the configuration is changed according to the rules induced by Lemma 7.1 (two particles can be added near a particle, two particles can die, if they are sufficiently near, or a particle can move of one step). The labels of the particles to add, to be erased or moved are given by $\underline{v}$ (for further explanations and a precise definition of the process see Chapter X of De Masi and Presutti (1991b)). Formula (6.20) follows from Lemma 7.1, by making some rough estimates to obtain upper bounds. In the case $a_p^q = 0$ we can have that one particle can move (next neighbor jump): this effect is taken into account in (6.18) (2 is replaced by $M + 1$ in the characteristic function).

If $q < M$ then $m(a^q) = 0$, so we have no v-functions left in these terms ($v_0^\varepsilon \equiv 1$). This induces us to consider separately the two cases $q = M$ and $q < M$. So we set

$$|v_{2n}^\varepsilon(\underline{x}, t)| \leq I_{q=M} + I_{q<M} \tag{6.21}$$

with obvious definition of $I_{q=M}$ and $I_{q<M}$ (following from (6.20)). Let us start with $I_{q=M}$.

In order to bound $I_{q=M}$ we use the fact that, given $t > s$, $N \in \mathbb{Z}^+$ and $f_{\leq s}$ a $\mathscr{F}_s$-measurable function ($\mathscr{F}_s$ is the sigma algebra of the events up to time s), we have (suppose that at time s a branching event has happened and then none)

$$\begin{aligned} &|\mathscr{E}_{\underline{x}}\{f_{\leq s}\mathbf{1}_{(|x_i(t)-x_k(t)|\leq N)}\}| \\ &\quad = |\mathscr{E}_{\underline{x}}\{f_{<s}\mathscr{E}_{\underline{x}}[\mathbf{1}_{(|x_i(t)-x_k(t)|\leq N)}\mathscr{F}_s]\}| \leq |\mathscr{E}_{\underline{x}}(f_{<s})|\frac{cN}{(\varepsilon^{-2}(t-s)+1)^{1-b}} \end{aligned} \tag{6.22}$$

(see formula (5.5)). By using (6.22) with $b > 0$ and Theorem 5.2 we have that

$$\begin{aligned} I_{q=M} \leq c\mathrm{e}^{2\alpha nt} \max_{a^M \in \mathscr{I}^M} \int_0^t \mathrm{e}^{2\alpha a_1^M s_1}\,\mathrm{d}s_1 \cdots \int_0^{s_{M-2}} \mathrm{e}^{2\alpha a_{M-1}^M s_{M-1}}\,\mathrm{d}s_{M-1} \\ \times \int_0^{s_{M-1}} \mathrm{d}s_M \left(\prod_{p=1}^M f_{a_p^M}(s_{p-1} - s_p)\right)(\varepsilon^{2-2b})^{m(a^M)}, \end{aligned} \tag{6.23}$$

where

$$f_a(t) = \begin{cases} \varepsilon^{2-2b}/t^{1-b}, & \text{if } a = 0, -1, \\ 1 & \text{if } a = +1. \end{cases} \tag{6.24}$$

Hence

$$I_{q=M} \leq c\mathrm{e}^{2\alpha nt} \max_{a^M \in \mathscr{I}^M} \mathrm{e}^{2\alpha t N(+1)} \varepsilon^{(2-2b)(N(0)+N(-1))} \varepsilon^{(2-2b)m(a^M)} I^\varepsilon(t)$$

in which $N(j) = \sum_{i=1}^M \mathbf{1}_{(a_i^M = j)}$ ($j = -1, 0, +1$) and $I^\varepsilon(t)$ is the multiple integral of the f_a factors (without the ε^{2-2b} factor that has been collected). $I^\varepsilon(t)$ can be easily bounded by some power of $|\log \varepsilon|$ (and so a fortiori by $\varepsilon^{-\zeta}$ for any $\zeta > 0$). Observe that $m(a^M) + N(0) + N(-1) = n + N(+1) + N(0)$. By the choice $t/|\log \varepsilon| = \tilde{\tau} < (2/3)\tau_c$ there is $\varrho > 0$ (choose it smaller than $2 - 2b$) such that $\mathrm{e}^{2\alpha\tilde{\tau}|\log \varepsilon|}\varepsilon^{2-2b} \leq \varepsilon^\varrho$. Hence

$$\begin{aligned} I_{q=M} &\leq c\varepsilon^{-\zeta}\mathrm{e}^{2\alpha nt}\varepsilon^{(2-2b)n} \max_{a^M \in \mathscr{I}^M} \varepsilon^{\varrho N(+1)}\varepsilon^{(2-2b)N(0)} \\ &\leq c\varepsilon^{-\zeta}\mathrm{e}^{2\alpha nt}\varepsilon^{(2-2b)n} \max_{a^M \in \mathscr{I}^M} \varepsilon^{\varrho(N(+1)+N(0))}. \end{aligned} \tag{6.25}$$

Observe that $N(+1) + N(0) = M - N(-1) \geq (M - n)/2$. Hence if we choose

$$M > \left[\frac{4n}{\varrho} + n\right] + 1 \tag{6.26}$$

we get

$$I_{q=M} \leq c(e^{2\alpha nt}\varepsilon^{3n})(\varepsilon^{-\zeta}(\varepsilon^{1-2b})^n) \leq c(e^{2\alpha t}\varepsilon^3)^n \tag{6.27}$$

for b sufficiently small. So $I_{q=M}$ largely matches with the estimate we want to obtain. Let us turn to $I_{q<M}$.

For sake of simplicity, let us first restrict to the case $q = n$. In this case there are only terms of the same type and the expectation value contains exactly n characteristic functions. We will call this term $I_{q=n}$. Now $a^n = (-1, \dots, -1)$, so

$$I_{q=n} \leq ce^{2\alpha nt} \sum_{\underline{v}\in\mathscr{S}^{a^n}} \int_0^t e^{-2\alpha s_1} ds_1 \cdots \int_0^{s_{n-1}} e^{-2\alpha s_n} ds_n$$
$$\times \mathscr{E}_{\underline{x}}\left\{\prod_{p=1}^n \mathbf{1}_{(|x_{i_p}(t-s_p) - x_{k_p}(t-s_p)| \leq M+1)}\right\}. \tag{6.28}$$

(Because of the properties of the stirring process, $\mathscr{E}$ can be exchanged with $\boldsymbol{E}^\varepsilon$ this time). Now we have to separate every integral (that can be separated) into two pieces: $\int_0^T ds + \int_T^t ds$, keeping fixed $T \in (0, t)$. It is clear that if $s_j < T$ for a certain j, all the bintegrals in $s_{j'}$ with $j' > j$ cannot be separated. We get

$$I_{q=n} \leq ce^{2n\alpha t} \sum_{\underline{v}\in\mathscr{S}^{a^n}} \Bigg\{\Bigg[\int_0^T ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{n-1}} ds_n + \int_T^t ds_1 \int_0^T ds_2 \cdots \int_0^{s_{n-1}} ds_n + \cdots$$
$$+ \int_T^t ds_1 \cdots \int_T^{s_{n-2}} ds_{n-1} + \int_0^T ds_n + \int_T^t ds_1 \cdots \int_T^{s_{n-1}} ds_n\Bigg]$$
$$\times \mathscr{E}_{\underline{x}}\left(\prod_{p=1}^n e^{-2\alpha s_p}\chi(p)\right)\Bigg\} \tag{6.29}$$

in which $\chi(p) = \mathbf{1}_{(|x_{i_p}(t-s_p) - x_{k_p}(t-s_p)| \leq M+1)}$. Now we estimate each term, noting that some of them can be easily bounded using only (5.5). For example, the last term between square brackets can be bounded by

$$ce^{2n\alpha t}\int_T^t ds_1 \cdots \int_T^{s_{n-1}} ds_n\, e^{-2\alpha Tn}\varepsilon^{(2-2b)n} \prod_i \frac{1}{(t-s_i)^{1-b}} \leq c' e^{2n\alpha t}(\varepsilon^{2-2b}e^{-2\alpha T})^n \tag{6.30}$$

in which we used (5.5). Once we fix $T/|\log \varepsilon| > \tau_c/3$, we get a good bound for this term. Analogously we can estimate the term preceding it in (6.29): in that case $\chi(n)$ can be estimated by means of (5.9), because the integrand is not singular. The problem arises when we deal with $\int_T^t \dots \int_T^{s_j-2} \int_0^T \int_0^{s_j} \dots \int_0^{s_{n-1}}$: we have to pay attention in estimating the last $j - n$ characteristic functions. The idea is that we cannot condition at time $t - s_{p-1}$ $(p \geq j)$ to estimate $\chi(p)$ (taking the supremum on the configurations at that time), but we have to trace back the evolution of the particles i_p, k_p to time $t - s_j$. This can be done using the well known technique of coupling stirring and independent particles (De Masi and Presutti, 1991b, Chapters VI and X). Slightly modifying the proof of Proposition 6.6.3 in De Masi and Presutti (1991b) to upgrade it to $d = 3$, we get that for all $a > 0$ (to be chosen small) and $K > 0$ (to be chosen large)

$$\boldsymbol{P}_{\underline{x},\underline{x}^0}(\|[\underline{x}(t) - \underline{x}] - [\underline{x}^0(t) - \underline{x}^0]\| \geq t^a) \leq ct^{-K} \tag{6.31}$$

in which $\boldsymbol{P}_{\underline{x},\underline{x}^0}$ is the coupled process stirring $(\underline{x}(t), \underline{x}(0) = \underline{x})$ and independent particles $(\underline{x}^0(t), \underline{x}^0(0) = \underline{x}^0)$. Formula (6.31) tells us that with probability $\approx (1 - \varepsilon^{2K})$ (K arbitrarily large), $\underline{x}(t)$ and $\underline{x}^0(t)$ are not farther than $\approx \varepsilon^{-2a}$ (a arbitrarily small). Taking this into account we get the right bound for each term in $I_{q=n}$ and it is easy to see that the *worst* one is exactly the first in the right-hand side of formula (6.29). Hence,

$$I_{q=n} \leq c(\xi)\varepsilon^{(3-\xi)n}e^{2n\alpha t}. \tag{6.32}$$

It is only a matter of direct inspection to see that in the general case $I_{m=m'}$, $m' \in \{n+1, \dots, M-1\}$, we get smaller terms (than is the case $m = n$). Roughly speaking this is because every time we iterate Lemma 7.1 we do not increase the order of the correlation function, we may get a small factor and we are left with a non-smaller order correlation function to estimate. Again the idea is to split every integral $(\int_0^T + \int_T^s)$, that can be split, and trace back the evolution of the particles to the moment in which they have been created. Again we have to couple our process with a proper *independent* process: this is done in Chapter X, Section 2 of De Masi and Presutti (1991b). Hence we prove that for all $m' \in \{n+1, \dots, M-1\}$ and for ε sufficiently small

$$I_{m=m'} \leq I_{m=n} \leq c(\xi)\varepsilon^{(3-\xi)n}e^{2n\alpha t} \tag{6.33}$$

and this completes the proof. The details are the same as in Chapter X in De Masi and Presutti (1991b); so we refer to it. □

Remark. There is no substantial reason for this method not to work in $d > 3$. It is very simple to see that Sections 3 and 4 work (and we already made some remarks about this), provided that there is a sequence $\{c_n\}$ such that

$$\|v_2^\varepsilon(t)\| \leq c_2 \frac{e^{2\alpha nt}\varepsilon^d}{|\log \varepsilon|^{d/2}}, \qquad \|v_{2n}^\varepsilon(t)\| \leq ce^{2\alpha nt}\varepsilon^{(d-\xi)n} \tag{6.34}$$

in which $t \geq \tau|\log \varepsilon|$ for a certain $\tau < \tau_c^d = d/2\alpha$. We need also to extend in a straightforward way the nonuniform estimates. The problem that arises is the following: $\|v_2^\varepsilon(t)\|$ (for example) is bounded as in (6.34) only if τ is sufficiently large (namely, $\tau > (d-2)/2\alpha$) and Theorem 5.2 gives an a priori good (infinitesimal) estimate only if $t \leq \tau|\log \varepsilon|$ with $\tau < 1/\alpha$. It is clear that for $d > 3$ we cannot use directly Theorem 5.2 to get an a priori estimate good enough to prove (6.34). On the other hand, it is clear that we can use Theorem 5.2 to get an estimate for a longer time and we can eventually iterate this procedure (for example, if $d = 4$ it is enough to do it once). Hence, we could prove (6.34) also for $d > 3$.

7. Technical lemmas

This section contains two Lemmas. The first one gives the integral expression for v_n that we used several times and it is only a matter of computation. The second one is extremely important, because it is the extension of the estimates on the *v-functions* to long times and all the theorems in Section 6 rely on it. The idea is that if we have the exact estimate on $v_{2n}(t)$, we subtract the *expected behavior* (see formula (7.8)) and we will remain with higher-order terms that can be quite easily estimated on a short time (proportional to $|\log\varepsilon|$).

Lemma 7.1 (Integral equation for v_{2n}). *For any $t > 0$ and $\underline{x} \in \mathscr{M}_{2n}$*

$$|v_{2n}^{\varepsilon}(\underline{x}, t)| \leq \int_0^t \exp(2\alpha n(t-s)) \boldsymbol{E}_{\underline{x}}^{\varepsilon}\{\mathscr{R}_n(\underline{x}(t-s), s)\}\, \mathrm{d}s, \tag{7.1}$$

where

$$\begin{aligned}\mathscr{R}_n(\underline{x}, \mathrm{t}) = 2\left(\frac{1}{d}\right)\sum_{l=1}^{d}\Big\{\sum_i \mathbf{1}_{(x_j \neq x_i \pm e_l\ \ \forall j \neq i)} |v_{2n+2}^{\varepsilon}(\underline{x}, x_i + e_l, x - e_l; t)| \\ + \sum_{i,k} \mathbf{1}_{(|x_i - x_k| \leq 2)} |v_{2n-2}^{\varepsilon} \underline{x}^{(i,k)}; t)| + \sum_{i,k} \sum_{b = \pm 1} \mathbf{1}_{(x_k = x_i + be_l)} [|v_{2n}^{\varepsilon}(\underline{x}, \mathrm{t})| \\ + \mathbf{1}_{(x_j \neq x_i - be_l,\ \forall j \neq k)} |v_{2n2}^{\varepsilon}(\underline{x}(k), \underline{x}_i - be_l; t)|]\Big\}.\end{aligned} \tag{7.2}$$

Proof. Using

$$\frac{\mathrm{d}}{\mathrm{d}t} v_{2n}^{\varepsilon}(\underline{x}, t) = \boldsymbol{E}_{\mu^{\varepsilon}}^{\varepsilon}\left(\boldsymbol{L}_{\varepsilon}\left(\prod_{x_i \in \underline{x}} \sigma(x_i)\right)\right). \tag{7.3}$$

After some algebra we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} v_{2n}^{\varepsilon}(\underline{x}, t) = (\varepsilon^{-2} + 4\gamma) L v_{2n}^{\varepsilon}(\underline{x}, t) + 2\alpha n v_{2n}^{\varepsilon}(\underline{x}, t) + \mathscr{Q}_n^{\varepsilon}(\underline{x}, t), \tag{7.4}$$

where

$$\begin{aligned}\mathscr{Q}_n^{\varepsilon}(\underline{x}, t) = \sum_{i=1}^{2n} \frac{1}{d} \sum_{l=1}^{d} \Big\{ -2\gamma^2 \mathbf{1}_{(x_i \pm e_l \neq x_j,\ \ \forall j \neq i)} v_{2n+2}^{\varepsilon}(\underline{x}, x_i + e_l, x - e_l; t) \\ + 2\gamma \sum_{k=1}^{2n} \frac{1}{d} \sum_{b = \pm 1} \mathbf{1}_{(x_i + be_l = x_k)} [v_{2n-2}^{\varepsilon}(\underline{x}(i,k); t) - v_{2n}^{\varepsilon}(\underline{x}; t) \\ - \gamma \mathbf{1}_{(x_i - be_l \neq x_j,\ \ \forall j \neq i)} v_{2n}^{\varepsilon}(\underline{x}(k), x_i - be_l) \\ - \frac{\gamma}{2} \sum_{m=1}^{2n} \mathbf{1}_{(x_i - e_l = x_m)} v_{2n-2}^{\varepsilon}(\underline{x}(k,m); t)]\Big\}.\end{aligned} \tag{7.5}$$

We have that for all n, all $\underline{x} \in \mathscr{M}^{\varepsilon}_{2n}$ and for all t

$$|\mathscr{Q}^{\varepsilon}_n(\underline{x}, t)| \leq \mathscr{R}^{\varepsilon}_n(\underline{x}; t) \tag{7.6}$$

(remember that $\gamma \in (1/2, 1)$). This proves the lemma. □

Theorem 7.2 (Extension of estimates up to the critical time (3D)). *If we know that there is $\bar{\tau} > \tau_c/3$ and a sequence $\{c_n\}$ such that for $t, \tilde{t}^- \equiv \bar{\tau}|\log\varepsilon| - \varepsilon^{\beta^*} \leq t \leq \bar{\tau}|\log\varepsilon| \equiv \tilde{t}$*

$$\| v_{2n}(t) \| \leq c_n e^{2\alpha n t} \frac{\varepsilon^{(3-b)n}}{|\log\varepsilon|^{\eta}} \equiv \Gamma^{\varepsilon, n}_{b,\eta}(t) \tag{7.7}$$

($b \geq 0, \eta \geq 0$) *for all $n \geq 1$, then there is $a > 0$ such that* (7.7) *works* (*replacing c with $2c$*) *for $t \in [\tilde{\tau}|\log\varepsilon|, (\tilde{\tau} + a)|\log\varepsilon|]$, provided $\tilde{\tau} + a < \tau_c - b/2\alpha$ and ε sufficiently small.*

Remark. For sake of precision, the statement of Theorem 7.2 is given taking care of the tiny interval of time that Theorem 5.1 does not cover. It is clear that once we have proved the statements of Theorems 6.2 and 6.3 for $t = \tilde{\tau}|\log\varepsilon|$ we have them also in a neighborhood of t of length ε^{β^*} (indeed $\tilde{\tau}$ can be chosen with a certain freedom). Hence we refer to this lemma by saying that we need to have (7.7) only for $t = \tilde{t}$ to be able to extend it to longer times.

Proof. Set $\tilde{t}^+ = \tilde{t} + a|\log\varepsilon|$. We observe that (7.7) would be trivial if $\tilde{\tau} + a > \tau_c - b/2\alpha$ (remember that $\|v_{2n}(t)\| \leq 1$) and the same is true also if $\tilde{\tau} + a \geq \tau_c - b/2\alpha$ in the case $\eta = 0$. Given $m \geq 1$, we set for $t \in [\tilde{t}, \tilde{t}^+]$

$$u^{\varepsilon}_{2m}(\underline{x}, t) = v^{\varepsilon}_{2m}(\underline{x}, t) - e^{2\alpha m(t-\tilde{t}^-)} \sum_{\underline{y}} \boldsymbol{P}^{\varepsilon}_{t-\tilde{t}^-}(\underline{x} \to \underline{y}) v^{\varepsilon}_{2m}(\underline{y}, \tilde{t}^-). \tag{7.8}$$

Clearly we have that

$$\| v_{2m}(t) \| \leq \| u^{\varepsilon}_{2m}(\cdot, t) \| + e^{2\alpha m(t-\tilde{t}^-)} \| v_{2m}(\tilde{t}^-) \| \tag{7.9}$$

and so we must prove that

$$\| u^{\varepsilon}_{2m}(\cdot, t) \| \leq \Gamma^{\varepsilon, m}_{b,\eta}(t) \tag{7.10}$$

for $t \in [\tilde{t}, \tilde{t}^+]$. In fact, if (7.10) holds, for $t \in [\tilde{t}, \tilde{t}^+]$ we have

$$\| v_{2m}(t) \| \Gamma^{\varepsilon, m}_{b,\eta}(t) + e^{2\alpha m t} e^{-2\alpha m \tilde{t}^-} \Gamma^{\varepsilon, m}_{b,\eta}(\tilde{t}^-). \tag{7.11}$$

By taking into account

$$e^{-2\alpha m(t-\tilde{t}^-)} \Gamma^{\varepsilon, m}_{b,\eta}(\tilde{t}^-) = \Gamma^{\varepsilon, m}_{b,\eta}(t) \tag{7.12}$$

we conclude the proof.

In order to prove (7.10) we note that by Lemma 7.1 and the hypothesis there is a constant $\varrho > 0$ such that

$$\|u_{2m}^{\varepsilon}(\cdot,t)\| \leq \varepsilon^{\varrho}\Gamma_{b,\eta}^{\varepsilon,m}(t) + c\mathrm{e}^{2\alpha mt}\int_{\tilde{t}^-}^{t}\mathrm{e}^{-2\alpha ms}\,\mathrm{d}s\Big\{\|u_{2m+2}^{\varepsilon}(\cdot,s)\|$$

$$+ \|u_{2m}^{\varepsilon}(\cdot,s)\|\max_{\underline{x},i,k}\boldsymbol{E}_{\underline{x}}^{\varepsilon}(\mathbf{1}_{(|x_i(t-s)-x_k(t-s)|=1)})$$

$$+ \|u_{2m-2}^{\varepsilon}(\cdot,s)\|\max_{\underline{x},i,k}\boldsymbol{E}_{\underline{x}}^{\varepsilon}(\mathbf{1}_{(|x_i(t-s)-x_k(t-s)|\leq 2)})\Big\}. \tag{7.13}$$

To derive this inequality we used

$$\|v_{2m}(t)\| \leq \|u_{2m}^{\varepsilon}(\cdot,t)\| + \Gamma_{b,\eta}^{\varepsilon,m}(t), \tag{7.14}$$

which followed from (7.8) and the observation (7.12). Formula (7.14) holds for any $m \geq 1$ and $t \in [\tilde{t}, \tilde{t}^+]$. By making some simple estimate (using mainly (5.5)) we replace the v-functions with u-functions, obtaining a small term (the first one on the right-hand side of (7.13)). The constant ϱ can be chosen strictly positive because $(\tau_c - (b/2\alpha)) - (\tilde{\tau} + a)$ and $\tilde{\tau} - (\tau_c/3)$ are strictly positive.

For $n \geq 1$ (actually the following formulae are sometimes meaningless for $n = 1$, but the modifications which make them work are trivial) we define

$$d_m = \varepsilon^{-\zeta}\max_{\tilde{t}\leq t\leq \tilde{t}^+}\mathrm{e}^{-2\alpha mt}\|u_{2m}^{\varepsilon}(\cdot,t)\|\cdot\begin{cases}\varepsilon^{-(3-b)m}, & \text{if } m \leq n,\\ \varepsilon^{-(3-b)n-\Theta(m-n)}, & \text{if } m > n.\end{cases} \tag{7.15}$$

We used and will use the constants chosen in the following way:

$$\zeta \in (0, \varrho), \tag{7.16a}$$

$$\Theta \in (2\alpha(\tilde{\tau} + a), 2\alpha(\tilde{\tau} + 3a/2)), \tag{7.16b}$$

$$a \in (0, a^*/3) \cap (0, 2\delta^*/3\alpha a) \cap (0, 2(\tau_c - \tau)/5). \tag{7.16c}$$

in which the $*$-constants are given by Theorem 5.1. It is clear that if we prove $d_n \leq 1$ we are through, because of (7.11) and (7.12). Some simple calculation starting from (7.13) and (7.15) lead to ($b' > 0$ is arbitrary)

$$d_m \leq c\{\mathrm{e}^{2\alpha\tilde{t}^+}\varepsilon^{3-b}d_{m+1} + \varepsilon^{2-2b'}d_m + \mathrm{e}^{-2\alpha\tilde{t}}\varepsilon^{2-2b'-3+b}d_{m-1} + \varepsilon^{\varrho-\zeta}\},$$
$$\text{if } m < n, \tag{7.17}$$

$$d_m \leq c\{\mathrm{e}^{2\alpha\tilde{t}^+}\varepsilon^{\Theta}d_{m+1} + \varepsilon^{2-2b'}d_m + \mathrm{e}^{-2\alpha\tilde{t}}\varepsilon^{2-2b'-3+b}d_{m-1} + \varepsilon^{\varrho-\zeta}\},$$
$$\text{if } m = n, \tag{7.18}$$

$$d_m \leq c\{\mathrm{e}^{2\alpha\tilde{t}^+}\varepsilon^{\Theta}d_{m+1} + \varepsilon^{2-2b'}d_m + \mathrm{e}^{-2\alpha\tilde{t}}\varepsilon^{2-2b'-\Theta}d_{m-1}$$
$$+ \varepsilon^{\varrho-\zeta+(3-b)-\Theta}\}, \quad \text{if } m > n \tag{7.19}$$

and b' derives from (5.5) (we replaced b' with $2b'$ to take care of factors $|\log \varepsilon|$). Furthermore,

$$d = \max_{1 \le n \le N} d_n \tag{7.20}$$

with N a suitable positive integer. Clearly d is bounded by the sum of d_n ($n = 1, \dots, N$) and to prove that d is bounded by 1 (for ε small enough) we need only to show that (i) the coefficients in (7.17)–(7.19) (multiplying the d's) go to zero with ε and (ii) that there is N such that d_N is smaller than 1.

(i) It is easy to see that these coefficients vanish with ε if the conditions (7.16) and the choice of $\tilde{t}$ are fulfilled (we have only to choose b' small enough).

(ii) It is sufficient to prove that there is N such that

$$\lim_{\varepsilon \to 0} \varepsilon^{-\zeta-(3-b)n+\Theta n} \max_{\tilde{t} \le t \le \tilde{t}'} \| v_{2N}(t) \| \mathrm{e}^{-2\alpha N t} \varepsilon^{-\Theta N} = 0. \tag{7.21}$$

In order to do this we observe that

$$\begin{aligned} v^{\varepsilon}_{2N}(\underline{x}, t) &= \boldsymbol{E}^{\varepsilon}_{\mu^{\varepsilon}}\left(\boldsymbol{E}^{\varepsilon}_{\mu^{\varepsilon}}\left(\prod_{i=1}^{2N} \{[\sigma(\boldsymbol{x}_i, t) - m(\varepsilon \boldsymbol{x}_i, t - \tilde{t}^-; \delta_{\tilde{\sigma}})] + m(\varepsilon \boldsymbol{x}_i, t - \tilde{t}^-; \delta_{\tilde{\sigma}})\} | \mathscr{F}_{\tilde{t}} \right)\right) \\ &= \boldsymbol{E}^{\varepsilon}_{\mu^{\varepsilon}}\left\{ \sum_{A \subset \{1, \dots, 2N\}} \left[v^{\varepsilon}_{|A|}(\underline{\boldsymbol{x}}_A, t - \tilde{t}^-; \delta_{\tilde{\sigma}}) \prod_{i \in A^c} m_{\varepsilon}(\varepsilon \boldsymbol{x}_i, t - \tilde{t}^-; \delta_{\tilde{\sigma}}) \right]\right\} \\ &\le \sum_{A \subset \{1, \dots, 2N\}} \left[\sup_{\lambda} | v^{\varepsilon}_{|A|}(\underline{\boldsymbol{x}}_A, t - \tilde{t}^-; \lambda)| \right] \left\{ \boldsymbol{E}^{\varepsilon}_{\mu^{\varepsilon}}\left[\prod_{i \in A^c} |m_{\varepsilon}(\varepsilon \boldsymbol{x}_i, t - \tilde{t}^-; \delta_{\tilde{\sigma}})| \right]\right\} \end{aligned} \tag{7.22}$$

in which we have taken the conditional expectation with respect to the σ-algebra $\mathscr{F}_{\tilde{t}^-}$ generated by the process till the time $\tilde{t}^-$. Furthermore $\delta_{\tilde{\sigma}}$ denotes the measure on X_ε supported by the configuration at time $\tilde{t}^-$ and m is the solution of the RD equation (2.4) with initial condition $\delta_{\tilde{\sigma}}$, i.e., of the system (4.2) with σ^* replaced by $\tilde{\sigma}$; $\underline{\boldsymbol{x}}_A$ is the vector $\boldsymbol{x}_i$ with $i \in A$.

First of all it is straightforward to show (by using the technique developed in Chapter 4 and using the estimates on the v_n up to time $\tilde{t}$) that for all $u > 0$ there is a c such that

$$\boldsymbol{P}^{\varepsilon}_{\mu^{\varepsilon}}(\| m(\cdot, t - \tilde{t}^-; \delta_{\tilde{\sigma}}) \| > \mathrm{e}^{\alpha t} \varepsilon^{-p+3/2}) \le c \varepsilon^u \tag{7.23}$$

in which p is an arbitrary strictly positive number. Now we observe that as $\| m_\varepsilon \|^m \ge \sup_{|\underline{x}|=m} | \prod_{i=1}^m m_\varepsilon(\varepsilon \boldsymbol{x}_i, t - \tilde{t}^-; \delta_{\tilde{\sigma}})|$ we have

$$\begin{aligned} \{ \| m_\varepsilon(\cdot, t - \tilde{t}^-; \delta_{\tilde{\sigma}}) \| > \mathrm{e}^{\alpha t} \varepsilon^{-p+3/2} \} \supset \Big\{ &\max_{\underline{x}: |\underline{x}| = m} \prod_{i=1}^m | m_\varepsilon(\varepsilon \boldsymbol{x}_i, t - \tilde{t}^-; \delta_{\tilde{\sigma}})| \\ &> (\mathrm{e}^{\alpha t} \varepsilon^{-p+3/2})^m \Big\}. \end{aligned} \tag{7.24}$$

Hence we can estimate the expectation in the last term of (7.22) (use (7.24) and the maximum principle) and we get

$$\|v_{2N}(t)\| \le C(N) \max_{m \le 2N} \max_{\underline{x}:|\underline{x}|=m} \varepsilon^{\delta^* m}[\mathrm{e}^{(2N-m)\alpha t}\varepsilon^{(-p+3/2)(2N-m)} + c(u)\varepsilon^u]. \tag{7.25}$$

Now choose p such that $2\alpha(\tilde{\tau}+a)+3-2p=\delta>0$ (choose $p=(3-2\alpha(\tilde{\tau}+a)/4$ so $\delta=(3-2\alpha(\tilde{\tau}+a))/2)$ and define $\delta'=\min(\delta^*,\delta)$. Set (for example) $u=2\delta'(2N-m)+1$ to get rid of the last term. Hence we get that there is c such that

$$\varepsilon^{-\zeta-(3-b)n+\Theta n} \max_{t\in[\bar{t},\bar{t}^+]} \|v_{2N}(t)\| \mathrm{e}^{-2\alpha N t}\varepsilon^{-\Theta N} \le c\varepsilon^{-n(\zeta+3-b-\Theta)}\mathrm{e}^{-2\alpha N\tilde{\tau}^-}\varepsilon^{-\Theta N}\varepsilon^{2\delta' N}. \tag{7.26}$$

Now observe that $\mathrm{e}^{-2\alpha N\tilde{\tau}^-}\varepsilon^{-\Theta N}\varepsilon^{2\delta' N} = \varepsilon^{(2\alpha\tilde{\tau}-\Theta+2\delta')N}$ and that $2\alpha\tilde{\tau}-\Theta+2\delta' > 2\delta' - 3\alpha a$. But we required $a<2\delta'/3$ (by (7.16c)), so the left-hand side of (7.26) is bounded by

$$c\varepsilon^{-n(\zeta+3-b-\Theta)}\varepsilon^{(2\delta'-3\alpha a)N}. \tag{7.27}$$

Choose

$$N > \frac{n(\zeta+3-b-\Theta)}{2\delta'-3a} \tag{7.28}$$

to conclude. □

Note. A further development in studying the onset of interfaces and their local structure (for the same dynamics considered in this paper) is contained in G. Giacomin's, *Interface formation and global spatial structure in a Reaction–diffusion model*, preprint Rutgers University, 1993. Phase separation has been studied also in the case of Glauber dynamics with Kac potential (see the works of A. De Masi, E. Orlandi, E. Presutti and L. Triolo, *Glauber evolution with Kac potential II: spinodal decomposition*, preprint, Rome, 1993). Moreover, further results on the motion by mean curvature for the process studied in this paper can be found in the works of M.A. Katsoulakis, P.E. Souganidis, *Interacting particle systems and generalized mean curvature evolution*, preprint, 1992.

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